Second-Order Differential Invariants for Some Extensions of the Poincaré Group and Invariant Equations

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Abstract

It is well-known that symmetry properties are extremely important for choosing differential equations which can be suitable for description of real physical processes.

We present functional bases of second-order differential invariants for various representations of some extensions of the Poincaré group and for a set of m scalar functions (e.g., for similarity and conformal groups). These results enable us to describe new classes of nonlinear multi–dimensional invariant equations and to simplify the problem of symmetry classification of second-order scalar partial differential equations.

0   Introduction

The concept of invariant is widely used in various domains of mathematics. In this paper we investigate differential invariants within the framework of symmetry analysis of differential equations.

Differential invariants and construction of invariant equations were considered by S. Lie [1] and his followers [2, 3]. Tresse [2] proved the theorem on existence and finiteness of a functional basis of differential invariants. However, there exist few papers devoted to construction of differential invariants for specific groups involved in mechanics and mathematical physics in explicit form.

Knowledge of differential invariants of a certain algebra or group facilitates classification of equations invariant with respect to this algebra or group. There are also some general methods for investigation of differential equations which need the explicit form of differential invariants for symmetry groups of these equations´ (see, e.g., [3, 4]).

A brief review of our investigation of second-order differential invariants for the Poincaré and Galilei groups is given in [5, 6]. Our results on functional bases of differential invariants are founded on Lemma about functionally independent invariants for the proper orthogonal group and for two n-dimensional symmetric tensors of order 2 [7].

We should like to stress that we consider functionally independent invariants of tensors but not irreducible ones, as in the classical theory of invariants.

Bases of irreducible invariants for the group $O(3)$ and three-dimensional symmetric tensors and vectors are adduced in [8].

DEFINITION 1. A maximal set of functionally independent invariants of order $r \leq l$ of the Lie algebra L is called a functional basis of the lth-order differential invariants for the algebra L.

We consider invariants of order 1 and 2, and we need the first and second prolongations of the operator

$$X = \xi^i \partial_{x_i} + \eta^r \partial_{u^r} \tag{1}$$

(see, e.g., [9–12])

$$\overset{1}{X} = X + \eta^r_i \partial_{u^r_i}, \qquad \overset{2}{X} = \overset{1}{X} + \eta^r_{ij} \partial_{u^r_{ij}},$$

the coefficients η^r_i and η^r_{ij} taking the form

$$\eta^r_i = (\partial_{x_i} + u^s_i \partial_{u^s}) \eta^r - u^r_k (\partial_{x_i} + u^s_i \partial_{u^s}) \xi^k,$$

$$\eta^r_{ij} = (\partial_{x_i} + u^s_j \partial_{u^s} + u^s_{jk} \partial_{u^s_k}) \eta^r_i - u^r_{ik} (\partial_{x_j} + u^s_j \partial_{u^s}) \xi^k.$$

While writing out lists of invariants, we shall use the following designations

$$u_a \equiv \frac{\partial u}{\partial x_a}, \quad u_{ab} \equiv \frac{\partial^2 u}{\partial x_a \partial x_b},$$

$$S_k(u_{ab}) \equiv u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k} u_{a_k a_1}, \tag{2}$$

$$S_{jk}(u_{ab}, v_{ab}) \equiv u_{a_1 a_2} \cdots u_{a_{j-1} a_j} v_{a_j a_{j+1}} \cdots v_{a_k a_1},$$

$$R_k(u_a, u_{ab}) \equiv u_{a_1} u_{a_k} u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k}.$$

Here and further we mean summation over repeated indices from 1 to n. In all lists of invariants, k takes values from 2 to n and j takes values from 0 to k. We shall not discern upper and lower indices with respect to summation: for all Latin indices

$$x_a x_a \equiv x_a x^a \equiv x^a x_a = x_1^2 + x_2^2 + \cdots + x_n^2.$$

When we calculate second prolongations with the usual Lie technique (see e.g. [9]), we imply that action of an operator of the form $X^{ab} \partial_{u_{ab}}$, where X^{ab} are some functions, is as follows:

$$X^{ab} \partial_{u_{ab}} (u_{cd} u_{cd}) = 2X^{ab} u_{ab}; \quad \partial_{u_{ab}} u_{cd} = \delta_{ac} \delta_{bd}.$$

With this assumption, $\partial_{u_{ab}} u_{ba} = 0$, $a \neq b$.

1 Differential Invariants for the Euclid Algebra

We consider the Euclid algebra $AE(n)$ defined by the basis operators

$$\partial_a \equiv \frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a \partial_b - x_b \partial_a. \tag{3}$$

Here and further, the letters a, b, c, d, when used as indices, take values from 1 to n, n being the number of space variables ($n \geq 3$).

The algebra $AE(n)$ is an invariance algebra for a wide class of multi-dimensional scalar equations involved in mathematical physics – the Schrödinger, heat, d'Alembert equations, etc.

In this section we shall explain in detail how to construct a functional basis of second-order differential invariants for the algebra $AE(n)$. This basis will be used further to find invariant bases for various algebras containing the Euclid algebra as a subalgebra.

1.1 MAIN RESULTS

THEOREM 1. *There is a functional basis of second-order differential invariants for the Euclid algebra $AE(n)$ with the basis operators (3) for a scalar function $u = u(x_1, ..., x_n)$ consisting of these $2n + 1$ invariants*

$$u, \qquad S_k(u_{ab}), \qquad R_k(u_a, u_{ab}). \tag{4}$$

THEOREM 2. *Second-order differential invariants of the algebra $AE(n)$ (1.1) for a set of scalar functions u^r, $r = 1, ..., m$, can be represented as functions of the following expressions:*

$$u^r, \qquad S_{jk}(u_{ab}^1, u_{ab}^r), \qquad R_k(u_a^r, u_{ab}^1). \tag{5}$$

Proofs of these theorems can be found in [7].

1.2 BASES OF INVARIANTS FOR THE EXTENDED EUCLID ALGEBRA
AND CONFORMAL ALGEBRA

The extended Euclid algebra $AE_1(n)$ for one scalar function is defined by the basis operators ∂_a, J_{ab} (3) and D depending on a parameter λ:

$$D = x_a \partial_a + \lambda u \partial_u \left(\partial_u = \frac{\partial}{\partial u} \right). \tag{6}$$

The basis of the conformal algebra $AC(n)$ consists of the operators ∂_a, J_{ab} (3), D (6) and

$$K_a = 2x_a D - x_b x_b \partial_a. \tag{7}$$

THEOREM 3. *There is a functional basis for the extended Euclid algebra that has the following form*
  (1) *when $\lambda \neq 0$:*

$$\frac{R_k(u_a, u_{ab})}{u^{k(1-2/\lambda)+1}}, \qquad \frac{S_k(u_{ab})}{u^{k(1-2/\lambda)}}; \tag{8}$$

  (2) *when $\lambda = 0$:*

$$u, \qquad \frac{R_k(u_a, u_{ab})}{(u_{aa})^k}, \qquad \frac{S_k(u_{ab})}{(u_{aa})^k} \qquad (k \neq 1); \tag{9}$$

  a functional basis for the conformal algebra has the following form:
  (1) *when $\lambda \neq 0$:*

$$S_k(\theta_{ab}) u^{k(2/\lambda - 1)}, \tag{10}$$

  (2) *when $\lambda = 0$:*

$$u, \ S_k(w_{ab})(u_a u_a)^{-2k} \qquad (k \neq n), \tag{11}$$

where

$$\theta_{ab} = \lambda u_{ab} + (1-\lambda)\frac{u_a u_b}{u} - \delta_{ab}\frac{u_c u_c}{2u},$$

$$w_{ab} = u_c u_c \left(u_{ab} + \frac{\delta_{ab}}{2-n}u_{dd}\right) - u_c(u_a u_{bc} + u_b u_{ac}),$$

(12)

δ_{ab} *being the Kronecker symbol.*

Proofs of Theorem 3 and all theorems below can be performed using the standard Lie algorithm [9, 10].

DEFINITION 2. Tensors θ_a and θ_{ab} of order 1 and 2 are called covariant with respect to some algebra $L = \langle J_{ab}, X_i \rangle$ if

$$X_i \theta_a = \sigma^i_{ab}\theta_b + \sigma^i \theta_a,$$

$$X_i \theta_{ab} = \rho^i_{ac}\theta_{cb} + \rho^i_{bc}\theta_{ac} + \rho^i \theta_{ab};$$

(13)

X_i are operators of the form (1), ρ^i, σ^i are some functions, σ^i_{ab}, ρ^i_{ab} are some skew-symmetric tensors.

It is easy to show that the expressions $S_k(\theta_{ab})$, $R_k(\theta_a, \theta_{ab})$, where θ_a, θ_{ab} are tensors covariant with respect to the algebra L, are relative invariants of this algebra.

The fact that θ_{ab} and w_{ab} (12) are covariant with respect to the conformal algebra $AC(n)$ can be verified by direct substitution of these tensors into the invariance conditions for the operators $\overset{2}{D}$ and $\overset{2}{K}_a$.

Note 1. When $\lambda \neq 0$, covariant tensors exist for $AC(n)$ of order 2 only; when $\lambda = 0$, the tensors w_{ab} (12) and u_a are conformally covariant but $S_k(w_{ab})$ and $R_k(u_a, w_{ab})$ are dependent.

THEOREM 4. *Second-order differential invariants for a vector function $u = (u^1, ...u^m)$ and for the algebra $AE_1(n) = \langle \partial_a, J_{ab}, D \rangle$, the operator D having the form*

$$D = x_a \partial_a + \lambda u^r \partial_{u^r}$$

(14)

with summation over r from 1 to m, can be represented as functions of the following expressions:

(1) *when $\lambda \neq 0$:*

$$\frac{u^r}{u^1} \quad (r = 2, ..., m), \quad \frac{S_{jk}(u^1_{ab}, u^r_{ab})}{(u^1)^{k(1-2/\lambda)}}, \quad \frac{R_k(u^r_a, u^1_{ab})}{(u^1)^{k(1-2/\lambda)+1}};$$

(2) *when $\lambda = 0$:*

$$u^r, \quad R_k(u^r_a, u^1_{ab}) \cdot (u^1_{aa})^{-k}, \quad S_{jk}(u^1_{ab}, u^r_{ab}) \cdot (u^1_{aa})^{-k}$$

(when $r = 1$ then $k \neq 1$);
the corresponding basis for the conformal algebra $AC(n) = \langle \partial_a, J_{ab}, D, K_a \rangle$, where

$$K_a = 2x_a D - x_b x_b \partial_a$$

has the following form:

(1) *when* $\lambda \neq 0$:

$$S_{jk}(\theta_{ab}^r, \theta_{ab}^1)(u^1)^{k(2/\lambda-1)}, \frac{u^r}{u^1}, \; R_k(\theta_a^r, \theta_{ab}^1)(u^1)^{k(2/\lambda-1)-1} \quad (r = 2, ..., m); \tag{15}$$

(2) *when* $\lambda = 0$:

$$u^r(r = 1, ..., m), (u_d^1 u_d^1)^{-2k} S_{jk}(w_{ab}^1, w_{ab}^r), \quad (u_d^1 u_d^1)^{-2k} R_k(u_a^r, w_{ab}^1) \quad (r = 2, ..., m) \tag{16}$$

(for the set of invariants $(u_d^1 u_d^1)^{-2k} S_k(w_{ab})$, k does not take the value n); the tensors θ_{ab}^r, w_{ab}^r are constructed similarly to (13) and

$$\theta_a^r = \frac{u_a^r}{u^r} - \frac{u_a^1}{u^1}.$$

1.3   DIFFERENTIAL INVARIANTS OF THE ROTATION ALGEBRA

The rotation algebra is defined by basis operators J_{ab} (3).

Second-order invariants of this algebra for m scalar functions u^r are constructed with x_a, u^r, u_a^r, w_{ab} similarly to invariants of the Euclid algebra.

THEOREM 5. *There is a functional basis of second-order differential invariants for the algebra $AO(n)$ that has the form*

$$u^r, \quad S_{jk}(u_{ab}^1, u_{ab}^r), \quad R_k(u_a^r, u_{ab}^1), \quad R_k(x_a, u_{ab}^1); \quad r = 1, ..., m;$$

the corresponding basis of invariants for the algebra $\langle J_{ab}, D \rangle$, where D is defined by (14), consists of the expressions

$$\frac{u^r}{u^1} \quad (r = 2, ..., m), \quad \frac{S_{jk}(u_{ab}^1, u_{ab}^r)}{(u^1)^{k(1-2/\lambda)}},$$

$$R_k(u_a^r, u_{ab}^1) \cdot (u^1)^{2k/\lambda-1-k},$$

$$R_k(x_a, u_{ab}^1) \cdot (u^1)^{2/\lambda(k-2)-k+1}, \qquad when \; \lambda \neq 0;$$

$$u^r, \; R_k(u_a^r, u_{ab}^1) \cdot (u_{aa}^1)^{-k},$$

$$S_{jk}(u_{ab}^1, u_{ab}^r) \cdot (u_{aa}^1)^{-k} \qquad (k \neq 1 \; when \; r = 1),$$

$$R_k(x_a, u_{ab}^1) \cdot (u_{aa}^1)^{2-k} \qquad when \; \lambda = 0.$$

A basis of invariants for the algebra $\langle J_{ab}, D, K_a \rangle$, when $\lambda \neq 0$, consists of the expressions (15) and

$$\frac{R_k(x_a, \theta_{ab}^1)}{x^2(u^1)^{(k-1)(1-2/\lambda)}}, \qquad k = 2, ..., n+1;$$

when $\lambda = 0$, it consists of the expressions (16) and

$$\frac{R_k(x_a, w_{ab}^1)}{x^2(w_{aa}^1)^{k-1}} \qquad (x^2 = x_a x_a).$$

2 Differential Invariants of the Poincaré and Conformal Algebra

In this section we consider differential invariants of the second order for a set of m scalar functions

$$u^r = u^r(x_0, x_1, ..., x_n), \qquad n \geq 3.$$

The Poincaré algebra $AP(1, n)$ is defined by the basis operators

$$
\begin{aligned}
p_\mu &= ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \\
J_{\mu\nu} &= x_\mu p_\nu - x_\nu p_\mu,
\end{aligned}
\tag{17}
$$

where μ, ν take values $0, 1, ..., n$; summation is implied over repeated indices (if they are small Greek letters) in the following way:

$$
\begin{aligned}
x_\nu x_\nu &\equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \cdots - x_n^2, \\
g_{\mu\nu} &= \mathrm{diag}\,(1, -1, ..., -1).
\end{aligned}
\tag{18}
$$

We consider x_ν and x^ν equivalent with respect to summation not to mix signs of derivatives and numbers of functions.

Quasilinear second-order invariants of the Poincaré algebra were described in [13].

THEOREM 6. *There is a functional basis of second-order differential invariants of the Poincaré algebra $AP(1, n)$ for a set of m scalar functions u^r consisting of*

$$m(2n + 3) + (m - 1)\frac{n(n + 1)}{2}$$

invariants

$$u^r, \qquad R_k(u_\mu^r, u_{\mu\nu}^1), \qquad S_{jk}(u_{\mu\nu}^r, u_{\mu\nu}^1).$$

In this section, everywhere $k = 1, ..., n + 1$; $j = 0, ..., k$; $r = 1, ..., m$.

For the extended Poincaré algebra $A\tilde{P}(1, n) = \langle p_\mu, J_{\mu\nu}, D \rangle$, where

$$D = x_\mu p_\mu + \lambda u^r p_{u^r} \tag{19}$$

($p_{u^r} = i(\partial/\partial u^r)$, summation over r from 1 to m is implied) the corresponding basis has the following form:
(1) when $\lambda = 0$:

$$u^r, S_{jk}(u_{\mu\nu}^r, u_{\mu\nu}^1) \cdot (u_{\alpha\alpha}^1)^{-k};$$

(2) when $\lambda \neq 0$:

$$\frac{u^r}{u^1}, \quad S_{jk}(u_{\mu\nu}^r, u_{\mu\nu}^1)(u^1)^{2k/\lambda - k - 1}, \quad R_k(u_\mu^r, u_{\mu\nu}^1)(u^1)^{k(2\lambda - 1)},$$

where S_{jk}, R_k are defined similarly to (2) and summation over small Greek indices is of the type (18).

For the conformal algebra $AC(1,n) = \langle p_\mu, J_{\mu\nu}, D, K_\mu \rangle$, where

$$K_\mu = 2x_\mu D - x_\nu x_\nu p_\mu$$

(D being the dilation operator (19)), the corresponding basis consists of the expressions

$$S_{jk}(\theta_\mu^r, \theta_{\mu\nu}^1) \cdot (u^1)^{k(2/\lambda - 1)}, \quad \frac{u^r}{u^1}, \quad R_k(\theta_\mu^r, \theta_{\mu\nu}^1) \cdot (u^1)^{k(2/\lambda - 1) - 1};$$

when $\lambda \neq 0$; $r = 2, ..., m$, there is no summation over r; conformally covariant tensors have the form

$$\theta_\mu^r = \frac{u_\mu^r}{u^r} - \frac{u_\mu^1}{u^1},$$

$$\theta_{\mu\nu}^r = \lambda u_{\mu\nu}^r + (1 - \lambda)\frac{u_\mu^r u_\nu^r}{u^r} - g_{\mu\nu}\frac{u_\beta^r u_\beta^r}{2u^r}$$

When $\lambda = 0$, the corresponding basis of invariants for the conformal algebra has the form

$$u^r, \quad S_{jk}(w_{\mu\nu}^r, w_{\mu\nu}^1) \cdot (u_\alpha^1 u_\alpha^1)^{-2k},$$

$$R_k(u_\mu^r, w_{\mu\nu}^1) \cdot (u_\alpha^1 u_\alpha^1)^{1-2k}, \qquad r = 2, ..., m;$$

the tensors $(w_{\mu\nu}^r)$,

$$w_{\mu\nu}^r = u_\alpha^r u_\alpha^r \left(u_{\mu\nu}^r - \frac{g_{\mu\nu}}{1 - n}u_{\beta\beta}^r\right) - u_\beta^r(u_\mu^r u_{\beta\nu}^r + u_\nu^r u_{\beta\mu}^r)$$

are conformally invariant (there is no summation over r).

Similarly to the results of Section 1.4, it is possible to construct invariants of the algebras $\langle J_{\mu\nu} \rangle$, $\langle J_{\mu\nu}, D \rangle$, $\langle J_{\mu\nu}, D, K_\mu \rangle$.

The obtained results allow us to construct new nonlinear multi-dimensional equations, e.g., the equation

$$\frac{u_\alpha u_\alpha}{1 - n}u_{\nu\nu} - u_\mu u_\nu u_{\mu\nu} = (u_\nu u_\nu)^2 F(u),$$

where F is an arbitrary function, is invariant under the algebra $AC(1,n)$, $\lambda = 0$.

There is another quasi-linear relativistic equation with rich symmetry properties

$$(1 - u_\alpha u_\alpha)u_{\mu\mu} - u_\alpha u_\mu u_{\alpha\mu} = 0,$$

that is the Born–Infeld equation. Symmetry and solutions of this equation were investigated in [11,14]. This equation is invariant under the algebra $AP(1, n+1)$ with the basis operators

$$J_{AB} = x_A p_B - x_B p_A, \qquad A, B = 1, ..., n+1; \quad x_{n+1} \equiv u.$$

Let us consider a class of equations

$$u_{\mu\nu} u_{\mu\nu} = F(u_{\mu\mu}, u_\mu u_\nu u_{\mu\nu}, u_\mu u_\mu, u).$$

It is evident that they are invariant with respect to the Poincaré algebra $AP(1,n)$ but a straightforward search for conformally invariant equations from this class with the standard Lie technique requires a lot of cumbersome calculations. Using of differential invariants turns this problem into one of elementary algebra, e.g., if $\lambda \neq 0$,

$$F - u_{\mu\nu}u_{\mu\nu} = -\frac{1}{\lambda}S_2(\theta_{\mu\nu}) + u^{2(1-2/\lambda)}\phi(S_1(\theta_{\mu\nu})u^{2/\lambda-1})$$

where $\theta_{\mu\nu}$ is of the form (1.24) and ϕ is an arbitrary function. Whence

$$F = u^{2(1-2/\lambda)}\phi\left(u^{2/\lambda-1}\left(u_{\mu\mu} - \frac{\lambda+n}{\lambda}\frac{u_\alpha u_\alpha}{u}\right)\right) -$$

$$\frac{1}{\lambda^2 u^2}(\lambda^2 + n^2)(u_\alpha u_\alpha)^2 - \frac{2(1-\alpha)}{\lambda u}u_\mu u_\nu u_{\mu\nu} + \frac{2u_{\mu\mu}u_\alpha u_\alpha}{\lambda u}.$$

It is useful to note that besides traces of matrix powers (3), one can utilize all possible invariants of covariant tensors $\theta^r_{\mu\nu}$, $w^r_{\mu\nu}$ to construct conformally invariant equations.

3 Differential Invariants of an Infinite-Dimensional Algebra

It is well-known that the simplest first-order relativistic equation – the eikonal or Hamilton equation

$$u_\alpha u_\alpha \equiv u_0^2 - u_1^2 - \cdots - u_n^2 = 0 \tag{20}$$

is invariant under the infinite-dimensional algebra $AP^\infty(1,n)$ generated by the operators [11, 15]

$$X = (b^{\mu\nu}x_\nu + a^\mu)\partial_\mu + \eta\partial_u, \tag{21}$$

$-b^{\mu\nu} = b^{\nu\mu}$, a^μ, η being arbitrary differentiable functions of u. Equation (20) is widely used in geometrical optics.

In this section, we describe a class of second-order equations invariant under the algebra (21).

It is easy to show that the tensors

$$\begin{aligned}\theta_\mu &= \frac{u_\mu}{(u_\alpha u_\alpha)^{1/2}}, \\ \theta_{\mu\nu} &= \frac{u_{\mu\nu}u_\beta u_\beta + u_{\beta\beta}u_\mu u_\nu - u_\beta u_{\beta\mu}u_\nu - u_\beta u_{\beta\nu}u_\mu}{(u_\alpha u_\alpha)^{3/2}}\end{aligned} \tag{22}$$

are covariant under the algebra $AP^\infty(1,n)$ (21).

THEOREM 7. *The equations of the form*

$$S_k(\theta_{\mu\nu}) = 0, \qquad k = 1, 2, ..., \tag{23}$$

S_k *being defined as (2), are invariant with respect to the algebra* $AP^\infty(1,n)$ *(21).*

The problem of description of all such equations is more difficult and we do not consider it here.

Convolutions of the form

$$R(\theta_\mu, \theta_{\mu\nu})$$

appear to be functionally dependent on the above.

Let us investigate in more detail a quasi-linear second-order equation of the form

$$u_\mu u_{\mu\nu} u_\nu - u_\mu u_\mu u_\alpha u_\alpha = 0. \tag{24}$$

THEOREM 8. *When $n \geq 2$, Equation (24) is invariant with respect to the algebra $A\tilde{P}^\infty(1,n)$ with generators of the form*

$$X + d(u)x_\mu p_\mu,$$

X is of the form (21), $d(u)$ is an arbitrary function of u.

Let us present another example of the $A\tilde{P}^\infty(1,n)$-invariant equation:

$$u_{\mu\nu} u_{\mu\nu} \cdot u_\beta u_\beta - 2u_\mu u_{\beta\nu} u_{\mu\nu} u_\beta - (\Box u)^2 u_\beta u_\beta = 0.$$

4   Conclusion

It is well-known that a mathematical model of physical or some other phenomenon must obey one of the relativity principles of Galilei or Poincaré. Speaking the language of mathematics, we can say that it means that equations of the model must be invariant under the Galilei or the Poincaré groups. Having bases of differential invariants for these groups (or for the corresponding algebras), we can describe all invariant scalar equations, or sort the invariant ones out of a set of equations.

Construction of differential invariants for vector and spinor fields presents more complicated problems. First-order invariants for a four-dimensional vector potential were found in [16]. The cases of spinor and multi-dimensional vector Poincaré–invariant equations and corresponding bases of invariants are still to be investigated.

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