**-Representations of the Quantum Algebra $U_q(sl(3))$

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Abstract

Studied in this paper are real forms of the quantum algebra $U_q(sl(3))$. Integrable operator representations of $\ast$-algebras are defined. Irreducible representations are classified up to a unitary equivalence.

1 Introduction

There is a quantum analog of the enveloping algebra $U_q(J)$, where $q \in \mathbb{C} \setminus \{0, \pm1\}$ is a parameter (see [1]) associated with each complex simple Lie algebra $J$. The quantum algebra $U_q(sl(3))$ is a $\mathbb{C}$-algebra generated by $k_i^{\pm1}, X_i, Y_i, i = 1, 2$, satisfying the relations:

$$[k_1, k_2] = 0, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i X_j = q^{a_{ij}} X_j k_i, \quad k_i Y_j = q^{-a_{ij}} Y_j k_i,$$

$$[X_i, Y_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}},$$

$$X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 = 0, \quad i \neq j,$$

$$Y_i^2 Y_j - (q + q^{-1}) Y_i Y_j Y_i + Y_j Y_i^2 = 0, \quad i \neq j,$$

where

$$a_{ij} = \begin{cases} -1/2, & i \neq j \\ 1, & i = j \end{cases} \quad \text{and} \quad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

It is natural for such algebras to study representations of their real forms. Some representations of $U_q(sl(3))$ were studied by different authors. In particular, a $\ast$-representation of $U_q(sl(2))$ was studied in [7]. All finite-dimensional representations of $U_q(sl(N))$, which are equivalent to a representation of the real form $su_q(N)$ (defined by the involution $X_i^* = Y_i$, $k_i^* = k_i$, for $q \in \mathbb{R}$ and $k_i^* = k_i^{-1}$, for $q \in \mathbb{T}$) were investigated in [1]. The paper [6] studied the so-called Harish-Chandra modules of $su_q(N,1)$, $q \in \mathbb{R}$, i. e., such representations that the spectra of $k_i$ belong to $q^{\mathbb{Z}/2}$, besides that, restriction of the representations to the subalgebra $su_q(N)$ is decomposed into an orthogonal sum of irreducible representations of $su_q(N)$ in such a way that each of them is contained in the decomposition once (a quantum analog of the representations of $su(N,1)$ which are integrable to the group $SL(N,\mathbb{R})$). Representations of another real form, the $\ast$-algebra $sl_q(3,\mathbb{R})$, $q \in \mathbb{R}$, defined by the involution $k_i^* = k_i^{-1}, X_i^* = X_i, Y_i^* = Y_i$, is described in [10].

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In this paper we study representations of real forms of $U_q(sl(3))$ by using a technique of semilinear relations developed in [9], [5]. Since the use of unbounded operators is necessary in each case, we give definitions of operator representations of $*$-algebras in a Hilbert space $H$. In accordance with these definitions, we describe all irreducible representation up to a unitary equivalence.

2 Object

The quantum algebra $U_q(sl(3))$ is the $\mathbb{C}$-algebra generated by $k_i$, $k_i^{-1}$, $X_i$, $Y_i$, $i = 1, 2$, satisfying the relations:

\[
[k_1, k_2] = 0, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1,
\]

\[
k_i X_j = q^{a_{ij}} X_j k_i, \quad k_i Y_j = q^{-a_{ij}} Y_j k_i,
\]

\[
[X_i, Y_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}},
\]

\[
X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 = 0, \ i \neq j,
\]

\[
Y_i^2 Y_j - (q + q^{-1}) Y_i Y_j Y_i + Y_j Y_i^2 = 0, \ i \neq j,
\]

where

\[
a_{ij} = \begin{cases} 
-1/2, & i \neq j \\
1 & i = j
\end{cases}
\]

and

\[
\delta_{ij} = \begin{cases} 
0, & i \neq j \\
1, & i = j
\end{cases}
\]

Remark 1 Transposition $k_i \leftrightarrow k_i^{-1}$ gives $U_q(sl(3)) \leftrightarrow U_{q^{-1}}(sl(3))$.

3 Real Forms ($*$-algebras)

Consider real forms of $U_q(sl(3))$. We will assume by definition that the real form of an algebra $A$ is determined by such an involution that:

1) it transforms a generator into a linear combination of generators,

2) axiom $(AB)^* = B^* A^*$ does not lead to a relation which is not a corollary of (1)–(3).

Consider nonisomorphic $*$-algebras, which are real forms of $U_q(sl(3))$. Set $t = q^{1/2}$.

Proposition 1 There are six real forms of $U_q(sl(3))$

\[
A_1: \quad k_i^* = k_i, \quad X_i^* = Y_i, \ t \in \mathbb{R}; \ k_i^* = k_i^{-1}, \ q \in \mathbb{T};
\]

\[
A_2: \quad k_i^* = k_i, \quad X_i^* = -Y_i, \ X_2^* = -Y_2, \ t \in \mathbb{R}; \ k_i^* = k_i^{-1}, \ q \in \mathbb{T};
\]

\[
A_3: \quad k_i^* = k_i, \quad X_i^* = -Y_i, \ t \in \mathbb{R}; \ k_i^* = k_i^{-1}, \ q \in \mathbb{T};
\]

\[
A_4: \quad k_i^* = k_i^{-1}, \quad X_i^* = X_i, \ Y_i^* = Y_i, \ t \in \mathbb{R}; \ k_i^* = k_i, \ q \in \mathbb{T};
\]

\[
A_5: \quad k_i^* = k_i^{-1}, \quad X_i^* = X_j, \ Y_i^* = Y_j, \ i \neq j, \ t \in \mathbb{R}; \ k_i^* = k_j, \ q \in \mathbb{T}, \ i \neq j;
\]

\[
A_6: \quad k_i^* = k_j, \quad X_i^* = Y_j, \ i \neq j, \ t \in \mathbb{R}; \ k_i^* = k_j^{-1}, \ i \neq j, \ q \in \mathbb{T};
\]

where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. 

\*

}\]}
For $q \in T$ $*$-algebras $A_1$, $A_2$, $A_3$ are isomorphic.

**Remark 2**

1) There are no $*$-structure for $U_q(sl(3))$ when $t \not\in R \cup T$.

2) Only $*$-algebras $A_1$, $A_2$, $A_3$, $A_3$ with $t \in R$: $A_4$, $A_6$ with $t \in T$ are $*$-Hopf-algebras (which means that involution agrees with comultiplication, counit and antipode). A complete description of $*$-Hopf algebras of the quantum algebras $U_q(J)$ ($J$ a is simple Lie algebra) is given, in particular, in [8].

### 4 $*$-Representations of $U_q(sl(3))$

To study the representations of different real forms of $U_q(sl(3))$ with using unbounded operators is necessary. Following [5], we give

**Definition 1** A collection of operators $k_i$, $X_i$, $Y_i$ is called a representation of $A_i$, $i = 1, 3$ in a Hilbert space $H$ if there exists a dense set $\Phi \subset H$ such that:

- a) $\Phi$ is invariant with respect to $k_i$, $X_i$, $Y_i$, $E(\Delta)$, $\Delta \in B(R^3)$, where $E(\cdot)$ is a joint resolution of indentity for the family of commuting selfadjoint operators ($k_i$, $X_i^*X_1$, $i = 1, 2$);
- b) $\Phi$ consists of bounded vectors for $k_i$, $X_i^*X_1$, $X_2^*X_2$ $i = 1, 2$;
- c) relations (1)–(3) hold on $\Phi$.

Under such a definition the technique of semilinear relation developed in [5] allows one to describe all the irreducible representations of the $*$-algebras up to a unitary equivalence. A detailed study of the representations of $*$-algebras $A_1$, $A_2$ is given in [5]. In particular, it was proved that there are representations of $A_2$ such that the spectrum of operator $k_i$, $i = 1, 2$ does not belong to $q^{Z/2}$ under such a definition. The following theorem gives a full description of the irreducible representations of $*$-algebra $A_3$.

**Theorem 1** For $q \in R$, $q > 1$, the $*$-algebra $A_1$ has the following irreducible representations:

a) $k_1 f_{m_1,m_2,m_3} = q^{m_1+(\beta+1+m_2-m_3)/2} f_{m_1,m_2,m_3}$,

$k_2 f_{m_1,m_2,m_3} = q^{m_3-m_2-(m_1+\delta+1)/2} f_{m_1,m_2,m_3}$,

$X_1 f_{m_1,m_2,m_3} = \sqrt{[m_1-m_3+1]_q [\beta+m_1+m_2+1]_q} f_{m_1+1,m_2,m_3}$,

$X_2 f_{m_1,m_2,m_3} = \sqrt{[m_2]_q [\delta+m_2]_q} \left( \prod_{r=0}^{m_1-1} \frac{[\beta+m_2+1]_q}{[\beta+m_2+r]_q} \right)^{1/2} \left( \prod_{r=0}^{m_1-1} \frac{[\beta+m_2+r-1]_q}{[\beta+m_2+r+1]_q} \right)^{1/2} f_{m_1,m_2-1,m_3}$,

$g(m_1,m_2,m_3) \left( \prod_{r=0}^{m_2-1} \frac{[\beta+m_2+m_3-r-1]_q}{[\beta+m_2+m_3-r+1]_q} \right)^{1/2} f_{m_1,m_2,m_3+1}$,

where
\[
g(m_1, m_2, m_3) = \begin{cases} 
\frac{[m_3 + 1]_q [\delta - m_3 - \beta]_q [m_1 - m_3]_q}{[\beta + m_3 + 1]_q} \times \\
\left( \prod_{r=0}^{m_3 - 2} \frac{[2 + r]_q}{[1 + r]_q} \right)^{1/2} \\
0, \\
\left( \prod_{r=0}^{m_1 - m_3 - 2} \frac{[2 + r]_q}{[1 + r]_q} \right)^{1/2}, \\
\frac{[m_3 + 1]_q [\delta - m_3 - \beta]_q [m_1 - m_3]_q}{[\beta + m_3 + 1]_q}, \\
\end{cases}
\]

with \(0 \leq m_3 \leq m_1, m_2 \geq 0, 0 \leq m_3 \leq s - 1, \beta \geq 0, \delta = \beta + s - 1;\)

b) \[
k_1 f_{m_1, m_2, m_3} = q^{-\beta - m_1 - (m_3 - m_2)/2} f_{m_1, m_2, m_3}, \\
k_2 f_{m_1, m_2, m_3} = q^{m_2 - m_3 + (m_1 + \delta)/2} f_{m_1, m_2, m_3}, \\
X_1 f_{m_1, m_2, m_3} = \sqrt{[m_1 - m_3]_q [\beta + m_1 + m_2]_q} f_{m_1 - 1, m_2, m_3}, \\
X_2 f_{m_1, m_2, m_3} = \sqrt{[m_2 + 1]_q [\delta + m_2]_q} \left( \prod_{r=0}^{m_1 - 1} \frac{[\beta + m_2 + r + 2]_q}{[\beta + m_2 + r + 1]_q} \right)^{1/2} \times \\
\left( \prod_{r=0}^{m_3 - 1} \frac{[\beta + m_2 + m_3 - r - 1]_q}{[\beta + m_2 + m_3 - r - 1]_q} \right)^{1/2} f_{m_1, m_2 + 1, m_3 + 1},
\]

where \(0 \leq m_3 \leq m_1, m_2 \geq 0, 0 \leq m_3 \leq s, \beta \geq 0, \delta = \beta + s + 1;\)

c) \[
k_1 f_{m_1, m_2, m_3} = q^{(\alpha - \beta + m_3 - m_2 - 1)/2 + m_1} f_{m_1, m_2, m_3}, \\
k_2 f_{m_1, m_2, m_3} = q^{m_2 - m_3 + (\delta - m_1 + 1)/2} f_{m_1, m_2, m_3}, \\
X_1 f_{m_1, m_2, m_3} = \sqrt{[\alpha + m_1 + m_3]_q [\beta - m_1 + m_2]_q} f_{m_1 + 1, m_2, m_3}, \\
X_2 f_{m_1, m_2, m_3} = \sqrt{[m_2 + 1]_q [\delta + \alpha + m_2 + 1]_q [\beta + m_2 + 1]_q} \times \\
\left( \prod_{s=0}^{m_1 - 1} \frac{[\beta + m_2 + s]_q}{[\beta + m_2 + 1 - s]_q} \right)^{1/2} \times \\
\left( \prod_{s=0}^{m_3 - 1} \frac{[\alpha + \beta + m_2 + s]_q}{[\alpha + \beta + m_2 + 1 - s]_q} \right)^{1/2} f_{m_1, m_2 + 1, m_3 + 1} + \\
\sqrt{[m_3]_q [\alpha + m_3 - 1]_q [\beta - \delta + m_3 - 1]} \times \\
\left( \prod_{s=0}^{m_1 - 1} \frac{[\alpha + m_3 + s]_q}{[\alpha + m_3 + s - 1]_q} \right)^{1/2} \times \\
\left( \prod_{s=0}^{m_2 - 1} \frac{[\alpha + \beta + m_3 + s]_q}{[\alpha + \beta + m_3 + s - 1]_q} \right)^{1/2} f_{m_1, m_2, m_3 - 1},
\]
where \( \alpha + \beta \geq 0, m_2 \geq 0, m_3 \geq 0, m_1 \in \mathbb{Z} \).

\( q \)-Representations of \( \ast \)-algebras \( A_1, A_5 \) provided that \( q > 1 \) can be also studied by the method of semilinear relations.

**Definition 2** A collection of operators \( k_i, X_i, Y_i \) \((k_i^\ast = k_i^{-1}, X_1 = X_1^\ast, Y_2 = Y_2^\ast, X_2, Y_1 \) are symmetric\) is called a representation of \( sl_q(3, \mathbb{R}) \), \( q \in \mathbb{R} \) in a Hilbert space \( H \) if there exists a dense set \( \Phi \subset H \) such that:

a) \( \Phi \) is invariant with respect to \( k_i, X_i, \) \( Y_i, E(\delta), \delta \in \mathcal{B}(\mathbb{R}^2), \) where \( E(\cdot) \) is a joint resolution of identity for the family of commuting selfadjoint operators \( X_1, Y_2; \)

b) \( \Phi \) consists of bounded vectors for the operators \( X_1, Y_2; \)

c) relations (1)–(3) hold on \( \Phi \).

**Theorem 2** For \( q \in \mathbb{R}, q > 1, \) the \( \ast \)-algebra \( sl_q(3, \mathbb{R}) \) has the following irreducible representations:

1) a one-dimensional: \( X_i = Y_i = 0, k_i = \pm 1, \pm i; \)

2) an infinite-dimensional: in \( l_2(\mathbb{Z}^2) = \{ f_{k,m}\} \)

\[
\begin{align*}
    k_1 f_{k,m} &= f_{k-2,m-1}, & k_2 f_{k,m} &= f_{k+1,m+2}, \\
    X_1 f_{k,m} &= c_1 q^k f_{k,m}, & Y_2 f_{k,m} &= c_2 q^m f_{k,m}, \\
    X_2 f_{k,m} &= \frac{1}{c_2 q^{2m}(q-q^{-1})^2} (f_{k+2,m} + f_{k-2,m} - q f_{k=2,m+4} - q^{-1} f_{k-2,m-4}), \\
    Y_1 f_{k,m} &= \frac{1}{c_1 q^m (q-q^{-1})^2} (f_{k,m+2} + f_{k,m-2} - q f_{k-4,m-2} - q^{-1} f_{k+4,m+2}),
\end{align*}
\]

\( c_1, c_2 \in \mathbb{R} = (-q^{1/2}, -1] \cup [1, q^{1/2}) \).

Definition of the representations of \( \ast \)-algebra \( A_5 \) for \( q > 1 \) and list of all irreducible representations see in [11].

**Remark 3** If \( q \in \mathbb{R} \) and the operator \( k_i, X_i, Y_i \) are bounded, then one can easily show that all irreducible representations of \( A_i, i = 4, 5, 6 \) are one-dimensional. The same is true for \( \ast \)-algebra \( A_6 \), when \( q \in \mathbb{R} \). It is a problem what are ”integrable” representations of such \( \ast \)-algebras for unbounded operators.
References


