Nonlinear Mathematical Physics 1997, V.4, N 1–2, 6–11.

Gauge Transformations for a Family of Nonlinear Schrödinger Equations

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Abstract

An enlarged gauge group acts nonlinearly on the class of nonlinear Schrödinger equations introduced by the author in joint work with Doebner. Here the equations and the group action are displayed in the presence of an external electromagnetic field. All the gauge-invariants are listed for the coupled nonlinear "Schrödinger-Maxwell" theory. Time-dependent gauge parameters result in additional terms of the type introduced by Kostin and Bialynicki-Birula and Mycielski, but Maxwell's equations for the (non-quantized) gauge-invariant electric and magnetic fields remain linear.

1 Nonlinear Schrödinger Equations

In earlier work Doebner and I introduced a family of nonlinear Schrödinger equations in order to interpret quantum-mechanically certain representations of infinite-dimensional algebras and groups [1–3]. We proposed these equations as candidates for describing quantum systems with dissipation. Let us now set them up in a slightly different but mathematically and physically useful form [4,5]. Put

$$\rho = \overline{\psi}\psi\,, \quad \hat{\mathbf{j}} = \frac{1}{2i}\left[\,\overline{\psi}\nabla\psi - (\nabla\overline{\psi})\psi\right]\,, \tag{1.1}$$

where $\psi(\mathbf{x},t)$ is a square-integrable, time-dependent wave function, ρ is the associated probability density in position as a function of time, and $\hat{\mathbf{j}}$ is a (non-gauge invariant) probability flux density. Introduce the real nonlinear functionals

$$R_1 = \frac{\nabla\cdot\hat{\mathbf{j}}}{\rho}, \quad R_2 = \frac{\nabla^2\rho}{\rho}, \quad R_3 = \frac{\hat{\mathbf{j}}^2}{\rho^2}, \quad R_4 = \frac{\hat{\mathbf{j}}\cdot\nabla\rho}{\rho^2}, \quad R_5 = \frac{(\nabla\rho)^2}{\rho^2}\,. \tag{1.2}$$

The total probability $\int\rho(\mathbf{x},t)\,d\mathbf{x}$ is conserved if ρ is the divergence of a vector field falling off at infinity. Noting that

$$\mathrm{Re}\left[\frac{\dot{\psi}}{\psi}\right] = \frac{1}{2}\,\frac{\dot{\rho}}{\rho}\,, \tag{1.3}$$

we can write our general nonlinear Schrödinger equation in a way that guarantees probability conservation, and accommodates from the start possible couplings with general external electric and magnetic potentials:

$$
\begin{aligned}
i\,\frac{\dot\psi}{\psi} \;=\; & i\left[\,\sum_{j=1}^{2}\nu_j R_j[\psi] \,+\, \frac{\nabla\cdot(\mathcal{A}(\mathbf{x},t)\rho)}{\rho}\,\right] + \\
& \left[\,\sum_{j=1}^{5}\mu_j R_j[\psi] \,+\, \frac{\nabla\cdot(\mathcal{A}_1(\mathbf{x},t)\rho)}{\rho} \,+\, \frac{\mathcal{A}_2(\mathbf{x},t)\cdot\hat{\mathbf{j}}}{\rho} \,+\, U(\mathbf{x},t)\right],
\end{aligned}
\tag{1.4}
$$

where: the ν_j $(j=1,2)$ and the μ_j $(j=1,\ldots,5)$ are real coefficients; $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ are distinct, external real-valued vector fields; and U is an external real-valued scalar field. The relationship of this equation to the usual, time-dependent linear Schrödinger equation

$$
i\hbar\dot\psi \;=\; \frac{[-i\hbar\nabla - (e/c)\mathbf{A}(\mathbf{x},t)]^2}{2m}\,\psi \,+\, e\,\Phi(\mathbf{x},t)\,\psi\,, \tag{1.5}
$$

where $\mathbf{A},\Phi$ are external electromagnetic potentials, is easily obtained from the expansion

$$
\frac{\nabla^2\psi}{\psi} \;=\; iR_1[\psi] + \frac{1}{2}R_2[\psi] - R_3[\psi] - \frac{1}{4}R_5[\psi]\,, \tag{1.6}
$$

and is given by

$$
\begin{aligned}
&\nu_1 = -\frac{\hbar}{2m}\,, \quad \nu_2 = 0\,, \quad \mathcal{A} = \frac{e}{2mc}\mathbf{A}\,,\\
&\mu_1 = 0\,, \quad \mu_2 = -\frac{\hbar}{4m}\,, \quad \mu_3 = \frac{\hbar}{2m}\,, \quad \mu_4 = 0\,, \quad \mu_5 = \frac{\hbar}{8m}\,,\\
&\mathcal{A}_1 = 0\,, \quad \mathcal{A}_2 = -\frac{e}{mc}\mathbf{A}\,, \quad U(\mathbf{x},t) = \frac{e}{\hbar}\,\Phi(\mathbf{x},t) \,+\, \frac{e^2}{2m\hbar c^2}\,\mathbf{A}^2.
\end{aligned}
\tag{1.7}
$$

The class of nonlinear equations that Doebner and I derived (in the absence of magnetic fields) allows the ν_j and μ_j to take arbitrary, real values (with ν_1 and μ_3 unequal to 0). Symmetries and reductions of such equations have been investigated by several authors [6,7]. An important goal is to understand whether and how the coefficients in these equations describe dissipative, diffusive, or irreversible processes in quantum mechanics.

2 Nonlinear Gauge Transformations

The nonlinear gauge group consists of certain local transformations of ψ that leave the probability density in position-space invariant. Its derivation and the justification for its interpretation as a group of gauge transformations have been discussed extensively elsewhere [4,5,8,9]. While nonlinear gauge transformations change the time-evolution, and can connect linear to nonlinear quantum theories, they do not change the physical content.

Write $\psi(\mathbf{x},t) = R(\mathbf{x},t)\exp iS(\mathbf{x},t)$, with R and S real. Let $\gamma,\Lambda \in \mathbf{R}$, with $\Lambda \neq 0$, and let $\theta(\mathbf{x},t)$ be a smooth real-valued function. Consider the group of transformations $\psi' = N[\Lambda,\gamma,\theta\,](\psi)$ defined by

$$
R' = R\,, \quad S' = \Lambda S + \gamma\ln R + \theta\,. \tag{2.1}
$$

Then the composition of two such transformations is given by

$$N[\Lambda_1,\,\gamma_1,\,\theta_1] \circ N[\Lambda_2,\,\gamma_2,\,\theta_2] \;=\; N[\Lambda_1\Lambda_2,\,\gamma_1+\Lambda_1\gamma_2,\,\theta_1+\Lambda_1\theta_2]\,, \tag{2.2}$$

and we have the semidirect product of the affine group in one dimension with the usual quantum-mechanical gauge group of $U(1)$-valued functions of $\mathbf{x}$ and t, acting nonlinearly in the Hilbert space. While N is not actually well-defined by Eq. (2.1) as a mapping (since S is only defined up to integer multiples of 2π), it is sufficient for the present interpretation that if ψ satisfies an equation in our class, there exists ψ' obeying a transformed equation. An appropriate selection of ψ' can always be made.

Under the gauge transformations of Eq. (2.1), it is straightforward to obtain

$$\begin{aligned} \rho' \;&=\; \overline{\psi'}\,\psi' \;=\; \rho\,,\\ \hat{\mathbf{j}}' \;&=\; \frac{1}{2i}\,[\,\overline{\psi'}\nabla\psi' - (\nabla\overline{\psi'})\psi'\,] \;=\; \Lambda\,\hat{\mathbf{j}} \,+\, \frac{\gamma}{2}\nabla\rho \,+\, \rho\nabla\theta\,. \end{aligned} \tag{2.3}$$

Now abbreviate Eq. (1.4) as $\dot{\psi}/\psi \;=\; ia+b$, and consider first the imaginary part a. Since $a = (1/2)\dot{\rho}/\rho$, it must be gauge-invariant: $a' = a$.

From this one easily determines how the coefficients and the field $\mathcal{A}$ transform under N:

$$\nu_1' = \frac{\nu_1}{\Lambda}\,, \quad \nu_2' = -\frac{\gamma}{2\Lambda}\nu_1 + \nu_2\,, \quad \mathcal{A}' = \mathcal{A} - \frac{\nu_1}{\Lambda}\,\nabla\theta\,. \tag{2.4}$$

As expected, the combination

$$\mathbf{j}^{\,gi} \;=\; \nu_1\hat{\mathbf{j}} \,+\, \nu_2\nabla\rho \,+\, \rho\mathcal{A} \tag{2.5}$$

is gauge-invariant: recall that $\mathbf{j}^{\,gi} \;=\; -\,\mathbf{J}/2$, where $\mathbf{J}$ is the usual gauge-invariant current obeying $\dot{\rho} \;=\; -\,\nabla\cdot\mathbf{J}$. The easiest way to calculate the remaining gauge-transformed quantities is to use the equation

$$b' = \Lambda b - \gamma a - \dot{\theta}\,, \tag{2.6}$$

obtained from transforming Eq. (1.4). By equating the coefficients of like terms, one finds

$$\begin{aligned} &\mu_1' = -\frac{\gamma}{\Lambda}\nu_1 + \mu_1\,, \quad \mu_2' = \frac{\gamma^2}{2\Lambda}\nu_1 - \gamma\nu_2 - \frac{\gamma}{2}\mu_1 + \Lambda\mu_2\,,\\ &\mu_3' = \frac{\mu_3}{\Lambda}\,, \quad \mu_4' = -\frac{\gamma}{\Lambda}\mu_3 + \mu_4\,, \quad \mu_5' = \frac{\gamma^2}{4\Lambda}\mu_3 - \frac{\gamma}{2}\mu_4 + \Lambda\mu_5\,, \end{aligned} \tag{2.7}$$

while the transformation laws for the external vector fields are

$$\begin{aligned} \mathcal{A}_1' \;&=\; \Lambda\,\mathcal{A}_1 \,-\, \gamma\,\mathcal{A} \,-\, \frac{\gamma}{2}\,\mathcal{A}_2 \,+\, \left(\frac{\gamma}{\Lambda}\,\nu_1 - \mu_1 + \frac{\gamma}{\Lambda}\,\mu_3 - \mu_4\right)\,\nabla\theta\,,\\ \mathcal{A}_2' \;&=\; \mathcal{A}_2 - \frac{2\mu_3}{\Lambda}\,\nabla\theta\,. \end{aligned} \tag{2.8}$$

The transformation of U is rather more complicated, and best understood by making use of some explicitly gauge-invariant parameters; we shall return to it shortly.

Next let us write a complete set of gauge invariants for the above system. Recall that the gauge-invariant quantities, *not* the original coefficients, must be the physically measurable entities in situations to be described by the nonlinear Schrödinger equations.

Since the 2-parameter subgroup indexed by Λ and γ acts on the 7-parameter space of coefficients, we must have 5 independent gauge-invariant parameters τ_j. We also have a gauge-invariant (magnetic) field $\mathcal{B}$:

$$\begin{aligned} &\tau_1 = \nu_2 - \frac{1}{2}\mu_1 , \quad \tau_2 = \nu_1\mu_2 - \nu_2\mu_1 , \quad \tau_3 = \frac{\mu_3}{\nu_1} , \quad \tau_4 = \mu_4 - \mu_1\frac{\mu_3}{\nu_1} , \\ &\tau_5 = \nu_1\mu_5 - \nu_2\mu_4 + \nu_2^2\,\frac{\mu_3}{\nu_1} , \quad \mathcal{B} = \nabla\times\mathcal{A} . \end{aligned} \tag{2.9}$$

But we can write in addition two new gauge-invariant combinations of the fields $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$, as follows:

$$\begin{aligned} \mathcal{A}_1^{gi} &= \nu_1\,\mathcal{A}_1 + \left(\frac{2\nu_2\mu_3}{\nu_1} - \mu_1 - \mu_4\right)\mathcal{A} - \nu_2\,\mathcal{A}_2 , \\ \mathcal{A}_2^{gi} &= \frac{\nu_1}{2\mu_3}\,\mathcal{A}_2 - \mathcal{A} . \end{aligned} \tag{2.10}$$

In discussions elsewhere it has been noted that τ_2 is related to the observed value of $\hbar/m$, and that nonzero values of τ_1 and τ_4, break time-reversal invariance while the parameters $\tau_3 \neq -1$ or $\tau_4 \neq 0$ break Galileian invariance. For time-reversal and Galileian invariant theories, $\iota_5 = (1/2)\tau_2 + \tau_5 \neq 0$ characterizes the deviation from linearizability. Here we observe further that the external fields $\mathcal{B}$ and $\mathcal{A}_2^{gi}$ change sign under time reversal, while the field $\mathcal{A}_1^{gi}$ does not. For the usual linear Schrödinger equation $\mathcal{A}_1^{gi}$ and $\mathcal{A}_2^{gi}$ are zero, and this value is not changed by gauging.

To return to our discussion of the scalar potential, it is helpful to rewrite Eq. (1.4) in terms of the gauge-invariant current $\mathbf{j}^{gi}$, the gauge-invariant fields $\mathcal{A}_1^{gi}$ and $\mathcal{A}_2^{gi}$, and (as far as possible) the gauge-invariant parameters τ_j. We obtain

$$\begin{aligned} i\,\frac{\dot\psi}{\psi} &= i\,\frac{\nabla\cdot\mathbf{j}^{gi}}{\rho} + \frac{1}{\nu_1}\left[\mu_1\,\frac{\nabla\cdot\mathbf{j}^{gi}}{\rho} + \tau_2\,\frac{\nabla^2\rho}{\rho} + \tau_3\,\frac{(\mathbf{j}^{gi})^2}{\rho^2} + (\tau_4 - 2\tau_1\tau_3)\,\frac{\mathbf{j}^{gi}\cdot\nabla\rho}{\rho^2} + \right. \\ &\left. \tau_5\,\frac{(\nabla\rho)^2}{\rho^2}\right] + \frac{1}{\nu_1}\left[\frac{\nabla\cdot(\mathcal{A}_1^{gi}(\mathbf{x},t)\rho)}{\rho} + 2\tau_3\,\frac{\mathcal{A}_2^{gi}(\mathbf{x},t)\cdot\mathbf{j}^{gi}}{\rho} - \hat{U}(\mathbf{x},t)\right] , \end{aligned} \tag{2.11}$$

where

$$\hat{U} = -\,\nu_1\,U + \tau_3\,\mathcal{A}^2 - \mu_4\,\nabla\cdot\mathcal{A} + 2\tau_3\,\mathcal{A}\cdot\mathcal{A}_2 - 2\,\tau_3\,\nu_2\,\nabla\cdot\mathcal{A}_2 . \tag{2.12}$$

In the case of the usual linear Schrödinger equation, $\hat{U} = (e/2m)\Phi$. Now Eq. (2.6), combined with the first transformation in Eqs. (2.7), gives us the rule

$$(\nu_1\,b + \mu_1\,a)' = (\nu_1\,b + \mu_1\,a) - \frac{\nu_1}{\Lambda}\,\dot\theta ; \tag{2.13}$$

applying this to Eq. (2.11) immediately yields the transformed scalar potential field,

$$\hat{U}' = \hat{U} + \frac{\nu_1}{\Lambda}\,\dot\theta . \tag{2.14}$$

Thus, $\mathcal{A}$ and $\hat{U}$ continue to transform according to the usual gauge transformations of electromagnetism, but with a coefficient ν_1/Λ. We have the gauge-invariant (electric) field,

$$\mathcal{E} = -\,\nabla\hat{U} - \frac{\partial\mathcal{A}}{\partial t} . \tag{2.15}$$

In short, it is fully consistent with any member of our family of nonlinear Schrödinger equations to introduce external electromagnetic fields. Defining $\mathbf{A}$ and Φ so that $\mathcal{A} = (e/2mc)\mathbf{A}$ and $\hat{U} = (e/2m)\Phi$, Maxwell's equations are unchanged for the corresponding gauge-invariant electromagnetic fields $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\nabla\Phi - (1/c)(\partial\mathbf{A}/\partial t)$. The coupled Schrödinger-Maxwell theory extends fully to the new class of nonlinear quantum-mechanical time evolutions, and the whole theory gauges. This analysis appears to confirm the correctness of our interpretation of nonlinear gauge transformations in relation to measurement [4, 5], when the quantum particle is interacting with external fields.

3 Time-Dependent Nonlinear Gauge Transformations

In joint work with Doebner and Nattermann [9], we demonstrate through careful analysis of the underlying physical assumptions that the proper class of nonlinear gauge transformations for quantum mechanics includes the case where Λ and γ depend on t (though these parameters cannot depend on $\mathbf{x}$). With $\gamma = \gamma(t)$ and $\Lambda = \Lambda(t)$, the coefficients ν_j and μ_j in Eq. (1.4) must of course be time-dependent. In addition, we show that the class of nonlinear Schrödinger equations must be extended to include two additional terms on the right-hand side of Eq. (1.4):

$$\alpha_1 \ln\rho + \alpha_2 S\,, \tag{3.1}$$

where α_1 and α_2 are real, time-dependent coefficients. These are respectively the nonlinear terms proposed by Bialynicki-Birula and Mycielski (BM) [10] and by Kostin (K) [11]. The relation of (BM) terms, (K) terms, and the terms in Eq. (1.4) with the separation property for N-particle hierarchies in quantum mechanics, has been examined in joint work with Svetlichny [12]. The present result sheds light on the proper physical interpretation.

Now Eq. (2.13) becomes

$$(\nu_1 b + \mu_1 a)' = (\nu_1 b + \mu_1 a) - \frac{\nu_1}{\Lambda}\dot{\theta} - \nu_1 \frac{\dot{\gamma}}{2\Lambda}\ln\rho - \nu_1 \frac{\dot{\Lambda}}{\Lambda} S\,, \tag{3.2}$$

which yields the transformations

$$\begin{aligned} \alpha_1' &= \Lambda\alpha_1 - \frac{\gamma}{2}\alpha_2 + \frac{1}{2}\left(\frac{\dot{\Lambda}}{\Lambda} - \dot{\gamma}\right), \\ \alpha_2' &= \alpha_2 - \frac{\dot{\Lambda}}{\Lambda}\,. \end{aligned} \tag{3.3}$$

New gauge-invariants are

$$\beta_1 = \nu_1\alpha_1 - \nu_2\alpha_2 + \nu_2\frac{\dot{\nu}_1}{\nu_1} - \dot{\nu}_2\,, \quad \beta_2 = \alpha_2 - \frac{\dot{\nu}_1}{\nu_1}\,. \tag{3.4}$$

Note that β_2 changes sign under time-reversal, while β_1 does not. In addition, Eq. (2.14) is changed; it becomes

$$\hat{U}' = \hat{U} + \frac{\nu_1}{\Lambda}\dot{\theta} + \frac{\nu_1}{\Lambda}\alpha_2\theta - \nu_1\frac{\dot{\Lambda}}{\Lambda^2}\theta\,, \tag{3.5}$$

so that we must also change the definition of the gauge-invariant electric field:

$$\mathcal{E} \;=\; -\nabla\hat{U} - \frac{\partial\mathcal{A}}{\partial t} \;-\; \beta_2\,\mathcal{A}\,. \tag{3.6}$$

Then Maxwell's equations remain linear, but the equation for $\nabla\times\mathcal{E}$ is modified in an interesting way when $\beta_2 \neq 0$. It becomes

$$\nabla\times\mathcal{E} \;=\; -\frac{\partial\mathcal{B}}{\partial t} \;-\; \beta_2\,\mathcal{B}\,, \tag{3.7}$$

while the other three Maxwell equations are unchanged. The physical interpretation of this system in relation to Kostin's equation is a subject of our current research.

Acknowledgments

I am grateful to W. Fushchych and the other organizers of the 1995 international conference in Kiev on Symmetry in Nonlinear Mathematical Physics for the invitation to talk about this work. I would like to thank H.-D. Doebner, P. Nattermann, and G. Svetlichny for interesting related discussions. Hospitality during the finalization of these results was provided by the Institut de Physique Théorique, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, and travel support by Rutgers University.

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