Two-Point Boundary Optimization Problem for Bilinear Control Systems

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Abstract

This paper presents a new approach to the optimization problem for the bilinear system

\[ \dot{x} = \{x, \omega\} \] (1)

based on the well-known method of continuous parametric group reconstruction using of its structure constants defined by the Brockett equation

\[ \dot{z} = \{z, \omega\}. \] (2)

Here \( x \) is the system state vector, \( \{\cdot, \cdot\} \) are the Lie brackets, \( z = \{x, y\} \), \( y \) is the vector of cojoint variables, \( \omega = A^{-1}z \) is the control vector, \( A \) is the inertia matrix.

The quadratic control functional has to reach an extremum at the optimal solution of the equation (2) and the boundary optimization problem is to find such \( z_0 \) that solution (2) makes evolution from the state \( x(t_0) = x_0 \) up to the final state \( x(t_1) = x_1 \) during the time delay \( T = t_1 - t_0 \). Therefore it is necessary to define a transformation group of the state space which is parametrized by components of the vector and then to solve the Cauchy problem for an arbitrary smooth curve joining \( x(t_0) \) with \( x(t_0) \).

Key words. Bilinear system, Lie group, optimization, boundary problem, structure constants.

1 Introduction

Optimization problem with a quadratic quality criterion for smooth a dynamic system

\[ \dot{x} = f(x, u), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \] (3)

in many important cases [1, 2] can be reduced to the bilinear form as follows: to find such a control \( u : R \rightarrow R^m, \quad u = u(t) \) for the system

\[ \dot{x} = \left( \sum_{\mu=1}^m H_\mu u_\mu \right) x, \] (4)

where \( H_\mu \) are matrices generating the Lie group \( G \) defined by \( f(x, u) \), that the state vector \( x \) varies from \( x_0 = x(t_0) \) to \( x_1 = x(t_1) \) and a loss functional riches a minimum. Brockett (1973) in [2] proposed instead of the equation for adjoint variable \( y \) another one for the commutator \( z = \{x, y\} \) as follows

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\[ \dot{z} = \{z, A^{-1}z\}, \quad (5) \]

where a matrix \( A \) can be expressed in terms of \( H_\mu \). Eliminating \( u_\mu \) on the basis on the Pontryagin maximum principle and expressing it via \( z \) yield the next two-point boundary problem. To find such \( z = z(t) \) that the system

\[ \dot{x} = \{x, A^{-1}z\} \quad \text{(6)} \]

in the force of (4) brings the state vector \( x \) from \( x(t_0) = x_0 \) to \( x(t_1) = x_1 \) during the time delay \( T = t_1 - t_0 \) which depends on coefficients of the quadratic loss functional.

An approach explained below gives global optimum in the case of a compact \( G \), otherwise a final compact approximation is necessary. Note that an usual linearization procedure applied to (3) gives only local optimum in all cases.

2 Main results

The optimization of bilinear system (6) is based on the well-known restoration method of continuous parametric group involving its structure constants defined from the Brockett equation (5).

Accordingly, we are to define a transformation group of the state space which is parametrized by the components of vector \( z_0 \). In the basis matched with the structure of the Lie algebra, we obtain that the equation (5) has the following form

\[ \dot{z}_\alpha = \sum_{\beta, \gamma=1}^{n} \frac{C_{\alpha}^{\beta\gamma}}{I_\gamma} z_\beta z_\gamma, \quad (7) \]

where \( C_{\alpha}^{\beta\gamma} \) are structural constants, \( I_\gamma \) are eigenvalues of the matrix \( A \).

The linear system, together with (7)

\[ \dot{x}_\alpha = \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \frac{C_{\alpha}^{\beta\gamma}}{I_\gamma} x_\beta z_\gamma, \quad (8) \]

is considered.

Under given \( z_0^j = z_j(t_0) \), \( x_0^j = x_j(t_0) \) one can represent a partial solution of a system (8) in the form

\[ x_\alpha(t) = \sum_{\beta=1}^{n} S_{\alpha\beta}(t, t_0; z_0^\gamma) x_\beta^0; \quad (9) \]

where \( S_{\alpha\beta}(t, t_0; z_0^\gamma) \) are elements of a fundamental matrix. The transformation (9) preserves a scalar product being a space rotation. If \( \tilde{x}_\alpha^0 = z_\alpha^0 \), the solution of a system (7) has the similar form

\[ z_\alpha(t) = \sum_{\beta=1}^{n} S_{\alpha\beta}(t, t_0; z_0^\gamma) z_\beta^0. \quad (10) \]
As fixed $z_0^\gamma (\gamma = 1, n)$, equation (7) defines variable coefficients of equation (8) and the fundamental matrix. Changing $t$, we obtain a one-parameter set of rotation of a space over a fixed point, the origin of coordinates. Fundamental matrices satisfy the group relations

$$
\sum_{\beta_1=1}^n S_{\alpha \beta_1} (t_2, t_1; z_0^\gamma) S_{\beta_1 \beta} (t_1, t_0; z_0^\gamma) = S_{\alpha \beta} (t_2, t_0; z_0^\gamma), \quad S_{\alpha \beta} (t, t_0; z_0^\gamma) = \delta_{\alpha \beta}
$$

and create a one-parameter Lie group according to time $t$. We note that system (7), (8) is invariant under the change of variables

$$
t = \tau T, \quad z_0^\gamma = \frac{\zeta_0^\gamma}{T}
$$

and, consequently, its fundamental matrix

$$
S_{\alpha \beta} \left( \tau T, \tau_0 T; \frac{z_0^\gamma}{T} \right) = S_{\alpha \beta} (t, t_0; z_0^\gamma)
$$

do not change.

If we take $\delta_{\beta_1 \beta}$ instead of $x_0^\alpha$, then after substitution (9) and (10) for (8) we obtain

$$
\frac{\partial}{\partial t} S_{\alpha \beta_1} (t, t_0; z_0^\gamma) = \sum_{\beta=1}^n \sum_{\gamma=1}^n C_{\alpha \beta}^\gamma I_\gamma S_{\beta \beta_1} (t, t_0; z_0^\gamma) \sum_{\gamma_1=1}^n S_{\gamma \gamma_1} (t, t_0; z_0^\gamma) z_0^\gamma.
$$

The variety of fundamental matrices $\| S_{\alpha \beta} (t, t_0; z_0^\gamma) \|$ under all possible $z_0^\gamma \in R^n$ and fixed $t = t_1 = t_0 + T$ forms a subgroup of the group $SO(n)$ i.e., the group of rotation of $n$-dimensional space.

By virtue of the change (13), it is sufficient to prove that the subgroup of $SO(n)$ is formed by matrices $S_{\alpha \beta} (t, t_0; z_0^\gamma)$ under every $t \in R$, $z_0^\gamma \in S^n$, where $S^n$ is a unit sphere in $R^n$:

$$
\sum_{\gamma=1}^n (z_0^\gamma)^2 = 1.
$$

Let $z_0^\gamma$ be directive cosines of a unit vector in $R^n$. Fixing $\zeta$ and changing $t$, we get the one-parameter set of matrices

$$
\{ \| S_{\alpha \beta} (t, t_0; z_0^\gamma) \| \}.
$$

Since $\sum_{\beta_1=1}^n S_{\alpha \beta_1} (t_0, t_1; z_0^\gamma) S_{\beta_1 \beta} (t_1, t_0; z_0^\gamma) = \delta_{\alpha \beta}$, then the variety of matrices (14) forms a group $G$ isomorphic to the group $SO(n)$. Choose $\vec{\zeta}_\mu$ as a unit vector with components $\zeta_\mu \gamma_1 = \delta_{\mu \gamma_1} (\mu, \gamma_1 = 1, n)$.

Then by (13) the infinitesimal matrices of corresponding one-parameter groups will have the following elements

$$
I^\mu_{\alpha \beta_1} = \lim_{t \to t_0} \frac{\partial}{\partial t} S_{\alpha \beta_1} (t, t_0; \vec{\zeta}_\mu) = \sum_{\beta=1}^n \sum_{\gamma=1}^n C_{\alpha \beta}^\gamma I_\gamma \delta_{\beta \beta_1} \sum_{\gamma_1=1}^n \delta_{\gamma \gamma_1} \delta_{\mu \gamma_1} = \frac{C_{\alpha \beta_1 \mu}}{I_\mu}.
$$
Compose a commutator and determine the structural constants of the group $G$

$$
\sum_{\delta=1}^{n} (I_{\alpha \beta} \Gamma_{\delta \beta}^{\gamma} - I_{\alpha \delta} \Gamma_{\beta \gamma}^{\delta}) = \frac{1}{I_{\gamma_1} I_{\gamma_2}} \sum_{\delta=1}^{n} (C_{\gamma_1 \delta} C_{\delta \gamma_2} - C_{\gamma_1 \delta} C_{\gamma_2 \delta}) = \frac{1}{I_{\gamma_1} I_{\gamma_2}} \sum_{\delta=1}^{n} (C^{\alpha \delta} C_{\delta \gamma_2}^{\gamma_1} - C^{\alpha \delta} C_{\gamma_2 \delta}^{\gamma_1})
$$

$$
C^{\alpha \beta}_{\gamma_1} C_{\delta \gamma_2}^{\gamma_1} = - \frac{1}{I_{\gamma_1} I_{\gamma_2}} \sum_{\delta=1}^{n} C_{\gamma_1 \delta} C_{\delta \gamma_2}^{\gamma_1} = \sum_{\gamma_3=1}^{n} \frac{I_{\gamma_3}}{I_{\gamma_1} I_{\gamma_2}} C_{\gamma_3 \gamma_2}^{\gamma_1} C^{\alpha \beta}_{\gamma_3} = \sum_{\gamma_3=1}^{n} A_{\gamma_3}^{\gamma_1 \gamma_2} I_{\alpha \beta}^{\gamma_3}.
$$

For them Jacobi’s identity is fulfilled

$$
\sum_{s=1}^{n} (A^{i s}_{p s} A^{j k}_{s} + A^{j s}_{p s} A^{k i}_{s} + A^{k s}_{p s} A^{i j}_{s}) = \sum_{s=1}^{n} \frac{I_{p}}{I_{s} I_{k} I_{k}} C^{i s}_{p s} C^{j k}_{s} I_{s} I_{k} I_{k} + \frac{I_{p}}{I_{s} I_{t} I_{k}} C^{k s}_{p s} C^{i j}_{s} I_{s} I_{k} I_{k} + \frac{I_{p}}{I_{s} I_{t} I_{k}} C^{k s}_{p s} C^{i j}_{s} I_{s} I_{k} I_{k} = 0. \quad (16)
$$

These infinitesimal operators form a dimensional Lie algebra. Its corresponding group is the $n$-parametrized Lie group $G$ with $z(t; \gamma; \vec{x}_0)$ as parameters.

A matrix $V_{a \beta}(z(t))$ of the adjoint representation of a group formed by fundamental matrices $\| S_{a \beta}(t_0 + T, t_0; z(t)) \|$ is determined according to [3] by the solution $W_{a \beta}$ of the following linear system of differential equations with constant coefficients

$$
dW_{a \beta}/dt = \delta_{a \beta} + \sum_{i=1}^{n} \sum_{j=1}^{n} Z^{i}_{a \beta} W_{j a}
$$

under the initial condition $W_{a \beta}(t_0; z(t_0)) = 0, (t = 0) \ (a, \beta, \gamma = 1, n)$, where $V^0_{a \beta}(z(t_0)) = W_{a \beta}(T, z(t_0))$. For restoration of a $n$-parameter group by means of structural constants, it is necessary to solve the Cauchy problem for a system (17).

According to [4] for the solution of an initial boundary-value problem, we need to solve also the second Cauchy problem for a system of linear equations in partial derivatives

$$
\partial \Gamma_{\alpha \beta}(\vec{\zeta})/\partial \vec{\zeta} = \sum_{\beta=1}^{n} \sum_{\mu=1}^{n} V^\mu_{a \beta}(\vec{\zeta}) \int^0_{\beta} (\vec{\zeta}) r^\beta(\vec{\zeta}) |_{\vec{\zeta}=0} = x_0, \quad \vec{\zeta} = \vec{\zeta}(S). \quad (18)
$$

For this, the trajectory connecting $\vec{x}_0$ and $\vec{x}_1$ in $R^n$ is given and a Riemann connexion is introduced

$$
\Gamma_{\alpha \beta}(\vec{\zeta}) = - \sum_{\mu=1}^{n} \int^0_{\beta} (\vec{\zeta}) V^\mu_{a \beta}(\vec{\zeta}).
$$

Then the Cauchy problem for equation (18) can be reduced to the definition $\vec{\zeta}(S), s \in [0, 1]$, from the equation

$$
\frac{dr^\alpha(S)}{dS} - \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \Gamma^\alpha_{\gamma \beta}(\vec{\zeta}) r^\gamma(S) \frac{dc^{\beta}}{dS} = 0; \quad \vec{\zeta}(0) = 0. \quad (19)
$$

The solution of a boundary-value optimization problem is obtained by integrating a system (6) with the initial condition $z(0) = \zeta(1)$. The approach proposed uses no iterative procedures and is applicable for solving the optimal control problems in a real time scale.
3 Conclusion

The analysis fulfilled above of the system with a multiplicative control demonstrated the following possibilities.

1. Construction of the Lie group representation basis with a minimum dimension.

2. Reduction of the two-point boundary optimization problem to Cauchy one for an auxiliary system which has to be integrated along a smooth fixed trajectory joining given points in the state space of the system.

3. Practically such a method is applicable for a real-time on-board control.

References


