An Analytic Approach to Torus Bifurcation in a Quasiperiodically Forced Duffing Oscillator

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Abstract

A weakly nonlinear quasiconservative Duffing oscillator under quasiperiodic forcing is studied with the help of an analytic expression for the complex Poincare mapping. This mapping is then used to analyze the quasiperiodic response of the oscillator and the different zones of various periodicity. This map plays the same significant role as the averages equations in the theory of a periodically forced Duffing oscillator.

A nonautonomous Duffing oscillator is described by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \delta \dot{x} + \alpha x^3 + x = f(t). \tag{1}$$

Study of a dynamical system under influence of periodic and quasiperiodic forcings is one of the most important problems of nonlinear science [5]. Two of the most important model systems that have been studied widely are Van–der–Pol [1] and Duffing oscillators [7]. Of late various methods have been proposed to analyze the various intricacies associated with such systems. Of course the numerical approach aided with a computer remains to be the ultimate arsenal to be used. Here we have studied a new form of the quasiperiodically forced Duffing oscillator with the help of the analytical approach which enables one to extract a behaviour of the response function of the system through an analytical expression of the Poincare map [6]. Of course, detailed theoretical characteristic properties of quasiperiodic forcing were analyzed by Scheurle [9] in an abstract way, whereas our approach is totally computational. The analytical map so constructed yields also detailed information regarding the different bifurcation zones and regions of periodic doubling. The essence of this method lies in the simplicity of the computational techniques involved. At this point, it will be quite justified to draw attention to the fact that there have been many attempts for setting up a discrete mapping corresponding to a continuous dynamical system. Among many such formulations, those of Ding [3], Grassman et al. [4], and Jensen et al. [8] are worth mentioning. We may point out in short the salient features of these to indicate that there are some differences from those of ours. In [3], the procedure is rather ad hoc and valid only for $s \rightarrow \alpha$, where s is the inverse of the relaxation time. On the other hand, we have used no such approximation in our computation reported below. In the reference [4], the situation was only periodic (not a quasiperiodic one as in

our case) and the procedure adopted was that of singular perturbation. Such an approach requires the scaling of space–time coordinates. No such scalling was used in our case. In the reference [8], the technique adopted was that of purely numerical simulation.

In this paper, we propose an analytical approach for nonlinear oscillations in oscillator (1) driven with the two–periodic force $f(t)$ and we study some bifurcations of two–dimensional tori in the oscillator phase space.

The Duffing oscillator under the quasiperiodic forcing is written as:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = -4\varepsilon\omega\left\{\delta_0\dot{x} + \frac{2}{3}\alpha\omega_0 x^3 + \omega Q\right\}, \tag{2}$$

$$Q = \sum f_n \sin(\omega_n t). \tag{3}$$

Here ω is the natural frequency of the oscillator, α_0 is the nonlinearity parameter, δ_0 is the dissipation constant, f_n and ω_n are the amplitudes and frequencies of the $(N+m+1)$ sinusoidal components of the external force $f(t)$, ε is a small dimensionless parameter denoting the difference between oscillator (2) and the usual harmonic oscillator. The double periodicity of $f(t)$ means that all ω_n can be presented as $\omega_n = \omega_0 + n\bar{\omega}$, $n = -M, -M+1, \ldots, N-1, N$, where ω_0 and $\bar{\omega}$ are independent basic frequencies.

Our considerations are based on the following main assumptions:

(i) oscillator (2) is weakly nonlinear and quasiconservative, $\varepsilon \ll 1$.

(ii) the number of sinusoidal components of $f(t)$ is large, $N, M \gg 1$.

We also suppose that the amplitudes f_n are all equal and $f = f_0$ for all $n = -M, \ldots, N$ and the frequency detuning $\bar{\omega}$ between neighbouring spectrum lines of the external perturbation is small, $\bar{\omega} = \varepsilon\Omega$. Furthemore, we set $(\omega_0 - \omega)/\omega = \varepsilon\triangle_0$. This means that a number of sinusoidal external oscillations drive simultaneously the oscillator in a resonant way.

If the right–hand side of (2) is totally absent, then we have the standard solution

$$x = u\cos\omega_0 t - v\sin\omega_0 t, \tag{4}$$

where u and v are constant. To include the effect of nonlinearity and perturbation we vary, u and v and obtain the following usual averaging proceedure [2].

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -\delta_0 u - \Delta_0 v - \alpha_0(u^2 + v^2)v + \sum f_n \cos(n\Omega t),$$
$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = \Delta_0 u - \delta_0 v + \alpha_0(u^2 + v^2)u, \tag{5}$$

where $\tau = \varepsilon\omega_0 t$. Let us now revert to the polar coordinates: $u = r\cos\phi$, $v = r\sin\phi$. Furthermore, we use the approximation

$$\lim_{N,M\to\alpha} \sum_{N=-M}^{N} f_n \cos(n\Omega t) = f_0 \sum_{-\alpha}^{\alpha} \cos(n\Omega t) = f_0 T \sum_{-\alpha}^{\alpha} \delta(\tau - kT) \tag{6}$$

with $T = 2\pi/\Omega$.

Now observe that when $f_0 = 0$, the variables (r, ϕ) obey

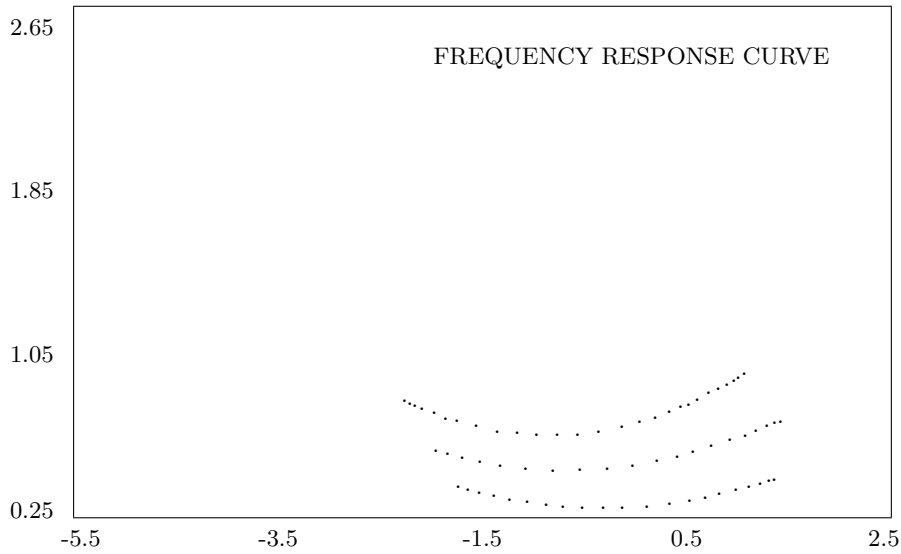$$\dot{r} = -\delta_0 r, \qquad \dot{\phi} = \Delta_0 + \alpha_0 r^2. \tag{7}$$

Fig.1. Frequency response curve. Variation of the detuning parameter Δ with respect to the intensity $E^{1/2}$ for $\alpha = 1$, $\delta = 0.5$. It clearly exhibits the discontinuous jump phenomena for several f values.

Whence the solution of (7) can be written as

$$r = r_0 \exp(-\delta_0 \tau), \qquad \phi = \phi_0 + \Delta_0 \tau - 2\alpha_0 \delta_0 r_0^2 \exp(-2\delta_0 \tau). \tag{8}$$

On the other hand, Eq.(5) leads us to a nonlinear system with a delta function forcing in a sequence. This enables us to pass over to a discrete mapping from the continuous system. The forcing term vanishes in the time interval (τ_k, τ_{k+1}), where $\tau = kT$. So we can not use solutions of the free system given in (8) to connect the values $r(\tau_k)$, $\phi(\tau_k)$ with $r(\tau_{k+1})$, $\phi(\tau_{k+1})$. So from the above we deduce ($z = u + iv$)

$$z(\tau_{k+1} + 0) = z(\tau_{k+1} - 0) + f_0 T, \quad \text{and} \tag{9}$$

$$z(\tau_{k+1} - 0) = z(\tau_k + 0) \exp(-\delta_0 T)\{\exp i\Psi\}, \tag{10}$$

where

$$\Psi = \Delta_0 T - 2\delta_0 \alpha_0 |z|^2 [\exp(-2\delta_0 T) - 1]. \tag{11}$$

So combining (9) and (10), we derive the mapping

$$z(\tau_{k+1} + 0) = z(\tau_k + 0) \exp(-\delta)\{\exp i\Psi_k\} + f, \tag{12}$$

where $\Psi_k = \Delta - 2\alpha |z(\tau_k + 0)|^2 [\exp(-2\delta) - 1]$, $\Delta = \Delta_0 T$, $\delta = \delta_0 T$, $\alpha = \alpha_0 \delta_0$, $f = f_0 T$.

We derive an analytic expression for the double Poincare map [10] of the weakly nonlinear Duffing equations under a quasiperiodic effect. This map is a one–dimensional complex map (or two–dimensional real map). Any period p periodic point of the map corresponds to a period pT periodic orbit of the averaged system (5). The latter corresponds again to a two–dimensional torus in the phase space of the initial model (2), with strongly incommensurate basic frequencies ω_0 and $\varepsilon \, \Omega \, \omega_0/p$. Therefore, one can study a double periodic motion of oscillator (2) by studying the fixed and periodic points of map (12). To show the
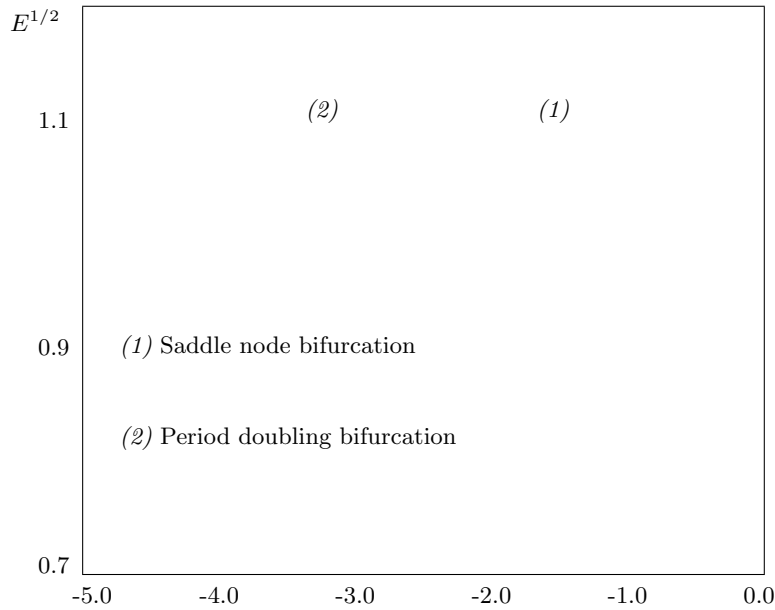
$E^{1/2}$

1.1   *(2)*   *(1)*

0.9   *(1)* Saddle node bifurcation

       (2) Period doubling bifurcation

0.7

    -5.0   -4.0   -3.0   -2.0   -1.0   0.0

Fig.2. Variation of the detuning parameter Δ with respect to the intensity $E^{1/2}$ for $\alpha = 1$, $\delta = 0.5$. Line (*1*) shows saddle node bifurcation and line (*2*) shows period doubling bifurcation.

correspondence between the discrete (12) and continuous ((2), (5)) models, we consider firstly the autonomous oscillation, i.e., suppose $f = 0$. It is easy to show that there is only one steady state of the map motion. The point $z = 0$ is fixed and stable because map (12) takes the form $\bar{z} = z \exp[i\Delta - \delta]$, $\Delta = (\Delta_0 T/2\pi)$, $\delta = (\alpha_0\delta_0)/2\pi$ near the origin of the complex plane.

Let us consider the case where $f \neq 0$. Map (12) contains two simple consequent actions over an arbitrary point z in the complex plane on each iteration. They are a nonlinear compression (or expansion) of the modulus of z and a shift along the real axis by the magnitude of f. The Jacobian of the map is a constant and $J = e^{(-2\delta)}$. Since $J \neq 0$ almost everywhere, the map is not area preserving.

The fixed point of the map satisfies the equation $\bar{z} = z$. This leads to the following equality for the magnitude $E = |z|^2$:

$$[1 - 2\exp(-\delta)\cos\{\Delta - 2\alpha E(\exp(-\delta) - 1)\} + \exp(-2\delta)]\,E = f^2, \tag{13}$$

where $E = |z^*|^2$, z^* being the fixed point. In the limit $T \to 0$, the above equation reduces to [6],

$$\left[f_0^2 + (\Delta_0 + 4\delta_0^2\alpha_0 I)^2\right] I = f_0^2, \tag{14}$$

where I represents the intensity of the forced oscillator. It is now interesting to note that Eq.(14) is nothing but the same one governing the jump phenomenon of a Duffing oscillator derived earlier by some other approaches. We now go back to (13) and obtain

$$\Delta = 2\alpha E(\exp(-\delta) - 1) + \cos^{-1}\left[\cosh\delta - (f^2 e^\delta)/(2E)\right]. \tag{15}$$

This equation is depicted graphically in Fig.1, where we have plotted $E^{1/2}$ vs detuning parameter Δ. It clearly exhibits the discontinuous jump phenomenon for several f values.
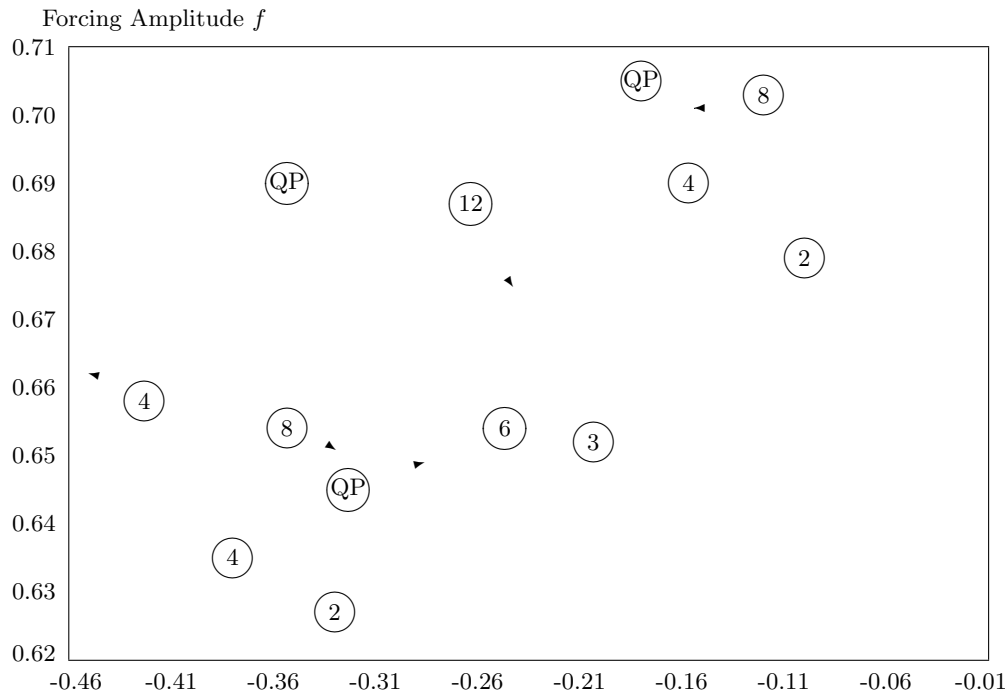
Fig.3. Variation of the detuning parameter Δ with respect to the intensity $E^{1/2}$ for $\alpha = 1$, $\delta = 0.5$. Different zones in the parametric plane pertaining to several types of periodicity are displayed. The numbers in the zones are the order of periodicity, and "QP" stands for the quasiperiodic state.

So in the first phase of our analysis, we have been able to deduce the behaviour of the response function in a completely different way.

Now so far the stability of the fixed point is concerned, we observe that the characteristic multipliers λ_1, λ_2 for fluctuations around z^*, given by (13), satisfy

$$\lambda^2 - S\lambda + j = 0, \tag{16}$$

where S is the trace of the linearized map. In our case,

$$S = \left[2\cos\left\{ \Delta - 2\alpha E(e^{-2\delta} - 1) \right\} + 4\alpha E \sin\left\{ \Delta - 2\alpha E(e^{-2\delta} - 1) \right\} (e^{-2\delta} - 1) \right] e^{-\delta}.$$

A fixed point loses stability when one or both of the $\lambda_{1,2}$ exit the unit circle in the complex plane. When the condition $|S/2| > J$ for some fixed point holds, then its characteristic multiplies are real. There are two different ways for losing stability in this case. The first of them occurs when $\max(\lambda_1, \lambda_2) = 1$, that is $S = 1 + J$, and it associates with the saddle node bifurcation. As for map (12), it occurs when

$$\Delta = 2\alpha E(e^{-2\delta} - 1) + \tan^{-1}\left[2\alpha E(e^{-2\delta} - 1) \right] + \cos^{-1}\left\{ (\cosh\delta)/[1 + 4\alpha^2(e^{-2\delta} - 1)E^2]^{1/2} \right\}.$$

This curve is shown in Fig.2 (curve (1)). The second way occurs when $\min(\lambda_1, \lambda_2) = -1$ or equivalently $S = -(1 + J)$. It corresponds to the period doubling bifurcation of the fixed point and it is realized under the condition

$$\Delta = 2\alpha E(e^{-2\delta} - 1) + \tan^{-1}\left[2\alpha E(e^{-2\delta} - 1) \right] - \cos^{-1}\left\{ (\cosh\delta)/[1 + 4\alpha^2(e^{-2\delta} - 1)E^2]^{1/2} \right\}.$$

This curve is shown in Fig.2 (curve (*2*)).

Finally, a bifurcation of the third type is possible when $J = 1$, i.e., when $\delta = 0$, the characteristic multipliers satisfy the equality $\lambda_1 \lambda_2^* = 1$ and the invariant curve arises in the neighbourhood of the fixed point in the z–plane.

The considered bifurcations of the fixed points of map (12) have a direct correspondence to the torus bifurcations in Eq.(2). The equation $S = +(1 + j)$ is the condition for the appearance of a stable three–dimensional torus whereas the equation $S = -(1 + j)$ corresponds to two–dimensional torus doubling bifurcations. Of course, to utilize these analytic results for detecting such bifurcations directly in the original system (2), one should assume that assumptions (i) and (ii) are fulfilled.

Lastly, the overall output of the map is shown in Fig.3, where different zones in the parametric plane pertaining to several types of periodicity are displayed. The number in the zones is the order of periodicity, and "QP" stands for the quasiperiodic state.

In the above analysis, we have shown that it is possible to derive a map in the case of a quasiperiodically forced Duffing oscillator, which can be used to study the nonlinear dynamics in a fruitful way.

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