

The Finite-Dimensional Moser Type Reduction of Modified Boussinesq and Super-Korteweg-de Vries Hamiltonian Systems via the Gradient-Holonomic Algorithm and Dual Moment Maps. Part I

A.K. PRYKARPATSKY*, O.E. HENTOSH* and D.L. BLACKMORE**

* Institute for Applied Problems in Mechanics and Mathematics of the National Academy of Sciences of Ukraine, 3b Naukova St., UA-290060 L'viv, Ukraine

** New Jersey Institute of Technology and Center of Applied Mathematics and Statistics, Newark, NJ-07102, U.S.A.

Abstract

The Moser type reductions of modified Boussinesq and super-Korteweg-de Vries equations upon the finite-dimensional invariant subspaces of solutions are considered. For the Hamiltonian and Liouville integrable finite-dimensional dynamical systems concerned with the invariant subspaces, the Lax representations via the dual moment maps into some deformed loop algebras and the finite hierarchies of conservation laws are obtained. A supergeneralization of the Neumann dynamical system is presented.

1 Introduction

This paper is concerned with finite-dimensional invariant subspaces of solutions to 2π -periodic nonlinear dynamical systems on infinite-dimensional functional manifolds being Lax integrable on them. The above-mentioned finite-dimensional reduction is realized effectively via the well-known Moser's procedure [1] using the gradient-holonomic algorithm [2] and the moment map technique, developed recently in [3, 4].

A modified Boussinesq hydrodynamic equation on M [11] on a functional space $M \subset \mathcal{C}^\infty(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{R}^2)$:

$$\left. \begin{aligned} du/dt &= 1/3v_x, \\ dv/dt &= u_{3x} + 6uu_x \end{aligned} \right\} = K[u, v]. \quad (1)$$

generates a smooth vector field $K : M \rightarrow T(M)$ on M considered as an infinite-dimensional manifold, being a completely integrable Hamiltonian system having a Lax-type representation

$$dl/dt = [l, (l^{3/2})_+], \quad (2)$$

where

$$l := \partial^3 + \frac{3}{4}(u\partial + \partial u) - \frac{i}{4}v, \quad i := \sqrt{-1}, \tag{3}$$

is a skew-symmetric differential operator in some Sobolev space $\mathcal{W}_\infty^{(3)}(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{C})$, the sign " + " denotes taking the positive part of an expression in the bracket.

The next dynamical system being interested for us is the Korteweg-de Vries super-Hamiltonian system [8, 12] on an infinite-dimensional 2π -periodic supermanifold $M^{1|1} \subset \mathcal{C}^\infty(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{R}^{1|1})$:

$$\left. \begin{aligned} du/dt &= -u_{3x} + 6uu_x - 12\xi\xi_x, \\ d\xi/dt &= -4\xi_{3x} + 6u\xi_x + 3u_x\xi \end{aligned} \right\} = K[u, \xi]. \tag{4}$$

The vector field $K : M^{1|1} \rightarrow T(M^{1|1})$ generates on $M^{1|1}$ a completely integrable dynamical system with the following Lax type representation:

$$(-D_\theta^3 + w)f(x, \theta) = 0, \tag{5}$$

where $w = (u - \lambda)\theta + \xi \in M^{1|1}$, $(x, \theta) \in \mathbf{R}^{1|1}$ are supervariables, $D_\theta := \partial/\partial\theta + \theta\partial/\partial x$ is the covariant superderivative, $\lambda \in \mathbf{C}$ is a spectral parameter and $f \in \mathcal{W}_\infty^{(3)}(\mathbf{R}^{1|1}/2\pi\mathbf{Z}; \mathbf{C}^{1|1})$ is the corresponding eigenfunction. Dynamical system (4), as will be shown below, reduces upon some invariant finite-dimensional submanifold to a new generalized Neumann type oscillatory super-Hamiltonian system on the sphere \mathbf{S}^N for any $N \in \mathbf{Z}_+$, being integrable via Liouville in quadratures.

2 The Moser type finite-dimensional reduction of a Boussinesq hydrodynamic system and its Lie-algebraic integrability

2.1 The Boussinesq hydrodynamic system (1) has an infinite hierarchy of conservation laws being written in the compact form as follows:

$$\gamma_j := \int_0^{2\pi} dx \operatorname{res} l^{j/3}, \quad j \in \mathbf{Z}_+, \tag{6}$$

where the operation "res" means the usual residue one on the space of pseudo-differential operators. From (5), one can find right away the simplest nontrivial conservation laws:

$$\gamma_1 = \int_0^{2\pi} dx u, \quad \gamma_2 = \int_0^{2\pi} dx v, \quad \gamma_3 = \int_0^{2\pi} dx uv. \tag{7}$$

To proceed further with an effective method devised in [4, 5] for the Moser-type nonlocal finite-dimensional reduction of the dynamical system (1) upon an invariant submanifold being alternative to previously developed in [6], we need to consider a space of eigenfunctions $\{f_j \in \mathcal{W}_\infty^{(3)}(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{C}) : j = \overline{0, N_\lambda}\}$ of the skew-symmetric Lax operator (3):

$$lf_j = \lambda_j f_j \Rightarrow f_{j,3x} + \frac{3}{2}u f_{j,x} + \frac{3}{4}u_x f_j - \frac{i}{4}v f_j = \lambda_j f_j, \tag{8}$$

where $\lambda_j \in i\mathbf{R}$, $j = \overline{0, N_\lambda}$, are some wholly imaging Blokh eigenvalues. The corresponding conjugated periodic eigenfunctions $\{f_j^* \in \mathcal{W}_\infty^{(3)}(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{C}) : j = \overline{0, N_\lambda}\}$ satisfy the adjoint to (8) equation as follows:

$$l^* f_j^* = -(lf_j)^* = \lambda_j f_j^* \tag{9}$$

owing to the skew-symmetry of operator (8).

It is well known that eigenvalues $\lambda_j \in \mathcal{D}(M)$, $j = \overline{0, N_\lambda}$, are smooth nonlocal invariant functionals on M . Hence, we can build, due to a Lax procedure, the following finite-dimensional invariant submanifold $M_N \subset M$:

$$M_N := \left\{ (u, v)^\tau \in M : \text{grad} \mathcal{L}_N[u, v] = 0 \right\}, \tag{10}$$

for any integer $N \in \mathbf{Z}_+$, where, by definition, the real-valued Lagrangian $\mathcal{L}_N \in \mathcal{D}(M)$ is taken in such a form:

$$\mathcal{L}_N := -\gamma_{N_\gamma+1} + \sum_{j=0}^{N_\gamma} a_j \gamma_j + \sum_{j=0}^{N_\lambda} b_j \lambda_j, \tag{11}$$

the integers $N_\gamma, N_\lambda \in \mathbf{Z}_+$ being fixed but arbitrary, $a_j \in \mathbf{R}$, $j = \overline{0, N_\gamma}$, $b_k \in i\mathbf{R}$, $k = \overline{0, N_\lambda}$, being some parameters. As was proven in [5], the manifold $M_N \subset M$ is in general a symplectic one, on which the vector fields d/dx and d/dt are Hamiltonian completely Liouville integrable dynamical systems.

2.2 Below we will be interested in the special case where $N_\gamma = 2$, $N_\lambda \in \mathbf{Z}_+$ are arbitrary. This gives, for some convenient choice of constants in (11), the following finite-dimensional submanifold

$$M_N = \left\{ (u, v)^\tau \in M : v = 3i \sum_{j=0}^{N_\lambda} (f_j^* f_{j,x} - f_{j,x}^* f_j), \quad u = \sum_{j=0}^{N_\lambda} f_j^* f_j \right\}, \tag{12}$$

where, obviously, the integer $N = 2N_\lambda + 2$, as easily follows from (11). Let us consider now the natural jet-coordinates upon the submanifold M_N locally imbedded in $\mathbf{C}^{2N_\lambda+2}$ as follows:

$$M_N \cong \left\{ (f_j, f_j^*; g_j = f_{j,x}, g_j^* = f_{j,x}^*; p_j = f_{j,xx}, p_j^* = f_{j,xx}^*)^\tau \in \mathbf{C}^{2N_\lambda+2} : \right. \\ \left. lf_j = \lambda_j f_j, l^* f_j^* = -\lambda_j f_j^*; \quad j = \overline{0, N_\lambda} \right\}. \tag{13}$$

The manifold M_N (12) carries a symplectic structure $\omega^{(2)}$ which can be retrieved from the following expression: $\omega^{(2)} := d\alpha^{(1)}$, where

$$d\mathcal{L}_N[u, v] = \langle \text{grad} \mathcal{L}_N[u, v], (du, dv)^\tau \rangle + d\alpha^{(1)}[u, v]/dx. \tag{14}$$

To find exactly the symplectic structure $\omega^{(2)}$ on the manifold M_N in coordinates (13) exactly, we need to determine the expression for $\mathcal{L}_N[u, v]$ as a local functional upon the extended infinite-dimensional manifold $\overline{M} := M \otimes (\mathcal{W}_\infty^{(3)})^{N_\lambda+2}$ as follows:

$$\mathcal{L}_N \longrightarrow \overline{\mathcal{L}}_N = \int_0^{2\pi} dx \overline{\mathcal{L}}_N[u, v; f_j, f_j^*]. \tag{15}$$

The condition $grad \bar{\mathcal{L}}_N[u, v; f_j, f_j^*] = 0$ defines in the manifold \bar{M} the next invariant finite-dimensional submanifold $\bar{M}_N \subset \bar{M}$:

$$\begin{aligned} \bar{M}_N := & \left\{ (u, v; f_j, f_j^* : j = \overline{0, N_\lambda})^\tau \in \bar{M} : v = 3i \sum_{j=0}^{N_\lambda} f_{j,x} f_j^* - f_{j,x}^* f_j, \right. \\ & u = \sum_{j=0}^{N_\lambda} f_j f_j^*, \quad f_{j,3x} - \frac{i}{4} v f_j + \frac{3}{4} u f_{j,x} + \frac{3}{4} (u f_j)_x - \lambda_j f_j = 0, \\ & \left. f_{j,3x}^* + \frac{i}{4} v f_j^* + \frac{3}{4} (u f_j^*)_x + \frac{3}{4} u f_{j,x}^* + \lambda_j f_j^* = 0 \right\}. \end{aligned} \tag{16}$$

Lemma 1 *The extended finite-dimensional jet-submanifold \bar{M}_N (16) is an invariant symplectic one with the following noncanonical symplectic structure:*

$$\begin{aligned} \bar{\omega}^{(2)} = & \sum_{j=0}^{N_\lambda} \left[df_j^* \wedge dp_j - dg_j^* \wedge dg_j + dp_j^* \wedge df_j + \right. \\ & \left. \frac{3}{4} (2u df_j^* \wedge df_j + f_j^* du \wedge df_j - f_j du \wedge df_j^*) \right]. \end{aligned} \tag{17}$$

◁ The proof is stemming from the following simple calculations due to the general theory of [5]:

$$\begin{aligned} d\bar{\mathcal{L}}_N[u, v; f_j, f_j^*] = & \\ & \langle grad \bar{\mathcal{L}}_N[u, v; f_j, f_j^*], (du, dv; df_j, df_j^* : j = \overline{0, N_\lambda})^\tau \rangle + d\bar{\alpha}^{(1)}[u, v; f_j, f_j^*]/dx, \end{aligned}$$

where

$$\bar{\alpha}^{(1)} = \sum_{j=0}^{N_\lambda} [f_j^* df_{j,2x} - f_{j,x}^* df_{j,x} + f_{j,2x} df_j + \frac{3}{4} (u f_j^* df_j - u f_j df_j^*)]. \tag{18}$$

Since the symplectic structure $\bar{\omega}^{(2)} := d\bar{\alpha}^{(1)}$, we can easily find from (18) that the expression (17) takes place exactly upon the submanifold $\bar{M}_N \subset \bar{M}$. ▷

The corresponding Hamiltonian functions for vector fields d/dx and d/dt on both the extended submanifold \bar{M}_N and submanifold M_N can be easily found by use of the following defining expressions:

$$\begin{aligned} \langle grad \bar{\mathcal{L}}_N[u, v; f_j, f_j^*], (u_x, v_x; f_{j,x}, f_{j,x}^* : j = \overline{0, N_\lambda})^\tau \rangle & := -d\bar{h}^{(x)}/dx, \\ \langle grad \bar{\mathcal{L}}_N[u, v; f_j, f_j^*], (u_t, v_t; f_{j,t}, f_{j,t}^* : j = \overline{0, N_\lambda})^\tau \rangle & := -d\bar{h}^{(t)}/dx. \end{aligned} \tag{19}$$

upon the submanifold $\bar{M}_N \subset \bar{M}$, and

$$\begin{aligned} \langle grad \mathcal{L}_N[u, v], (u_x, v_x)^\tau \rangle & = -dh^{(x)}/dx, \\ \langle grad \mathcal{L}_N[u, v], (u_t, v_t)^\tau \rangle & = -dh^{(t)}/dx \end{aligned} \tag{20}$$

upon the submanifold $M_N \subset M$. From (19), one can find, for example, that

$$\bar{h}^{(x)} = \sum_{j=0}^{N_\lambda} (p_j^* g_j - p_j g_j^* + \lambda_j f_j f_j^*) + \frac{i}{4} v \left(\sum_{j=0}^{N_\lambda} f_j^* f_j - u \right). \tag{21}$$

If we reduce the symplectic structure (17) upon the submanifold $M_N \subset \overline{M}_N$ putting $u = \sum_{j=0}^{N_\lambda} f_j^* f_j$, $v = 3i \sum_{j=0}^{N_\lambda} (f_j^* g_j - f_j g_j^*)$, we will get the symplectic structure $\omega^{(2)}$ on the submanifold M_N in jet-coordinates as follows:

$$\omega^{(2)} = \sum_{j=0}^{N_\lambda} \left\{ df_j^* \wedge dp_j - dg_j^* \wedge dg_j + dp_j^* \wedge df_j + \frac{3}{4} d \left[\left(\sum_{k=0}^{N_\lambda} f_k^* f_k \right) (f_j^* df_j - f_j df_j^*) \right] \right\}. \tag{22}$$

Thus, we can formulate the following theorem:

Theorem 1 *The vector field d/dx on the submanifold $M_N \subset M$ is a Hamiltonian one with respect to the noncanonical symplectic structure (22) and the Hamiltonian function*

$$h^{(x)} = \sum_{j=0}^{N_\lambda} (p_j^* g_j - p_j g_j^* + \lambda_j f_j^* f_j), \tag{23}$$

that is $i \frac{d}{dx} \omega^{(2)} = -dh^{(x)}$ on the submanifold M_N . The same is also true for the vector field d/dt with a Hamiltonian function $h^{(t)} \in \mathcal{D}(M_N)$ easily being found from the following determining evolution equations for the eigenfunctions $f_j, f_j^* \in \mathcal{W}_\infty^{(3)}(\mathbf{R}/2\pi\mathbf{Z}; \mathbf{C})$, $j = 0, N_\lambda$:

$$df_j/dt = i(l^{3/2})_+ f_j = i(f_{j,xx} + u f_j), \quad df_j^*/dt = -i((l^{3/2})_+ f_j)^* = -i(f_{j,xx}^* + u f_j^*). \tag{24}$$

As a natural consequence of the above proven results, we can state the following assertion about exact solutions to the hydrodynamic Boussinesq system (1).

Theorem 2 *All 2π -periodic orbits of the completely integrable Hamiltonian vector field d/dx upon the submanifold $M_N \subset M$ with respect to the symplectic structure $\omega^{(2)}$ (22) and the Hamiltonian function $h^{(x)}$ (23) consist of 2π -periodic solutions (8) and (9), giving rise to a set of exact 2π -periodic solutions to the hydrodynamic Boussinesq system (1) via the following formulas:*

$$u = \sum_{j=0}^{N_\lambda} f_j f_j^*, \quad v = 3i \sum_{j=0}^{N_\lambda} (f_j^* g_j - f_j g_j^*)$$

for each $N_\lambda \in \mathbf{Z}_+$, being fixed arbitrarily.

2.3 We proceed now to the description of the vector field d/dx on the submanifold $M_N \subset M$ by means of the Lie-algebraic approach developed recently in [3, 4] and the gradient-holonomic algorithm described in [2]. Let us consider the monodromy matrix $S(x; \lambda)$ as a special periodic solution to the following commutator Novikov-Lax equation:

$$dS(x; \lambda)/dx = [L[u, v; \lambda], S(x; \lambda)] \tag{25}$$

for all $\lambda \in \mathbf{C}$, where the matrix $L[u, v; \lambda]$ has the form:

$$L[u, v; \lambda] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ iv/4 - 3u_x/4 + \lambda & -3u/2 & 0 \end{pmatrix}. \tag{26}$$

The matrices L and S , as is seen, belong to some Lie subalgebra $\tilde{\mathcal{G}}$ of the affine Lie algebra $sl(3; \mathbf{C}) \otimes \mathbf{C}(\lambda, \lambda^{-1})$. The corresponding analytical loop subgroup \tilde{G}_+ acts on the complex matrix eigenfunction's manifold $(F, Q, R) \in M_{N,3}^3$ as follows:

$$\begin{aligned} F &\rightarrow F_g := \text{res}_{\lambda=\infty} \left(\frac{1}{\lambda + A} F g^{-1}(\lambda) \right), \\ Q^\tau &\rightarrow Q_g^\tau := \text{res}_{\lambda=\infty} \left(g(\lambda) Q^\tau \frac{1}{\lambda + A} \right), \\ R^\tau &\rightarrow R_g^\tau := \text{res}_{\lambda=\infty} \left(g(\lambda) R^\tau \frac{1}{\lambda + A} \right), \end{aligned} \tag{27}$$

where $g(\lambda) \in \tilde{G}_+$ is arbitrary, the "pole"-matrix $A := \text{diag}\{\lambda_j \in i\mathbf{R} : j = \overline{0, N_\lambda}\}$. The matrix manifold $M_{N,3}^3$ carries the symplectic structure $\Omega^{(2)}$, modelling structure (17) on \overline{M}_N , being invariant under the action (27):

$$\Omega^{(2)} := -Sp(dF \wedge dQ^\tau) + \frac{3}{4} d\left\{ Sp(FR^\tau) Sp[(dF)R^\tau - F(dR^\tau)] \right\} \tag{28}$$

together with the following matrix reductions: $R := F^* P_1$ and $Q := F^* \sigma$, where

$$P_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{29}$$

are the projection and intertwining matrices, correspondingly, giving rise to the symplectic structure (17).

Denote by $\tilde{\mathcal{G}}_+$ the Lie algebra of the loop group \tilde{G}_+ . The group action (27) engenders a moment map $J_N : M_{N,3}^3 \rightarrow \tilde{\mathcal{G}}_+^*$, which is defined as follows:

$$J_N(F, Q, R; \lambda)(X(\lambda)) := H_X(F, Q, R). \tag{30}$$

Here, by definition,

$$-dH_X(F, Q, R) := i_{K(F,Q,R)} \Omega^{(2)}, \quad (dF/dt, dQ/dt, dR/dt)^\tau := K(F, Q, R)$$

is a vector field on the matrix manifold $M_{N,3}^3$ defined as follows:

$$\begin{aligned} dF/dt &:= -\text{res}_{\lambda=\infty} \left(\frac{1}{\lambda + A} F X(\lambda) \right), \\ dQ^\tau/dt &:= \text{res}_{\lambda=\infty} \left(X(\lambda) Q^\tau \frac{1}{\lambda + A} \right), \\ dR^\tau/dt &:= \text{res}_{\lambda=\infty} \left(X(\lambda) R^\tau \frac{1}{\lambda + A} \right). \end{aligned} \tag{31}$$

From (30) and (31), we can state that the following lemma is true:

Lemma 2 *The $\tilde{\mathcal{G}}_+$ -action (27) is Hamiltonian, with an equivariant moment map $J_N : M_{N,3}^3 \rightarrow \tilde{\mathcal{G}}_+^*$ defined by*

$$J_N(F, Q, R; \lambda) = Q^\tau \frac{1}{\lambda + A} F + \frac{3}{2} R^\tau \frac{Sp(R^\tau F)}{\lambda + A} F. \tag{32}$$

◁ Since from (28) and (31) one obtains

$$H_X(F, Q, R) = \text{res}_{\lambda=\infty} Sp \left(\left(Q^\tau \frac{1}{\lambda + A} F + \frac{3}{2} R^\tau \frac{Sp(R^\tau F)}{\lambda + A} F \right) \cdot X(\lambda) \right), \quad (33)$$

the result (32) immediately holds for the \tilde{G}_+ -action (27) on the manifold $M_{N,3}^3$, having taken into account that $\tilde{\mathcal{G}}_+^* \cong \tilde{\mathcal{G}}_-$ with respect to the following symmetric and invariant scalar product on the Lie subalgebra $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$:

$$(a, b) := \text{res}_{\lambda=\infty} Sp(a \cdot b)$$

for all $a, b \in \tilde{\mathcal{G}}$. ▷

Reducing the moment map (32) upon the invariant submanifold $M_N \subset M_{N,3}^3$ by means of substitutions $R = F^* P_1$, $Q = F^* \sigma$, we obtain that $\Omega^{(2)}|_{M_N} = \omega^{(2)}$ and

$$J_N(f_j, g_j, p_j; \lambda) = \sum_{j=0}^{N_\lambda} \frac{1}{\lambda + \lambda_j} \begin{pmatrix} f_j^* p_j + \frac{3}{2} u f_j^* f_j & -f_j^* g_j & f_j^* f_j \\ g_j^* p_j + \frac{3}{2} u g_j^* f_j & -g_j^* g_j & g_j^* f_j \\ p_j^* p_j + \frac{3}{2} u p_j^* f_j & -p_j^* g_j & p_j^* f_j \end{pmatrix}^+. \quad (34)$$

Here the sign "+" above denotes the standard Hermitian conjugation in the adjoint space \mathcal{G}^* .

In other way, the monodromy matrix $S(x; \lambda)$ reduced via Moser [1] upon the invariant submanifold $M_N \subset M$ takes the following form:

$$\begin{aligned} S(x; \lambda)|_{M_N} &:= S_N(f_j, g_j, p_j; \lambda) = \\ &\sum_{j=0}^{N_\lambda} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} p_j f_j^* + \frac{3}{2} u f_j f_j^* & -g_j f_j^* & f_j f_j^* \\ p_j g_j^* + \frac{3}{2} u f_j g_j^* & -g_j g_j^* - \frac{1}{2} u f_j^* f_j & f_j g_j^* \\ p_j p_j^* + \frac{3}{2} u f_j p_j^* & -g_j p_j^* & f_j - \frac{1}{2} u f_j f_j^* \end{pmatrix} + \\ &\sum_{j=0}^{N_\lambda} \frac{u}{\lambda - \lambda_j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{4}(f_j g_j^* + f_j^* g_j) & 0 & 0 \end{pmatrix} + \frac{3}{4} \sum_{j=0}^{N_\lambda} \frac{u_x}{\lambda - \lambda_j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f_j g_j^* + f_j^* g_j & 0 & 0 \end{pmatrix} + \quad (35) \\ &\sum_{j=0}^{N_\lambda} \frac{v}{\lambda - \lambda_j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3i}{4} f_j g_j^* + \frac{i}{2} f_j^* g_j & f_j f_j^* & 0 \end{pmatrix} + \sum_{j=0}^{N_\lambda} \frac{v_x}{\lambda - \lambda_j} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{i}{2} f_j f_j^* & 0 & 0 \end{pmatrix} + \end{aligned}$$

$$\sum_{j=0}^{N_\lambda} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} 0 & 0 & 0 \\ 2\lambda_j f_j f_j^* & 0 & 0 \\ 3\lambda_j f_j g_j^* + \lambda_j f_j^* g_j & 2\lambda_j f_j f_j^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ u & 0 & 0 \\ \frac{1}{2}u - \frac{i}{6}v & 0 & 0 \end{pmatrix}.$$

It is easy to see that the monodromy matrix (35) can not be obtained right away from the moment map (34) because of the inequality $J_N^+(f_j, g_j, p_j; \lambda^*) \neq S_N(f_j, g_j, p_j; \lambda)$ for all $\lambda \in \mathbf{C}$. This means that the preceding group action (27) must be realized by means of a group action of some affine Lie group $\delta\tilde{\mathcal{G}}_+$ isomorphic to the previous affine Lie group $\tilde{\mathcal{G}}^+$.

Therefore, we need to change definition (30) as follows:

$$\delta J_N(F, Q, R; \lambda)(\delta X(\lambda)) := H_{\delta X}(F, Q, R) = H_X(F, Q, R), \tag{36}$$

where $\delta : \tilde{\mathcal{G}} \rightleftharpoons \delta\tilde{\mathcal{G}}$ is some still unknown deformative isomorphism of affine Lie algebras $\tilde{\mathcal{G}}$ and $\delta\tilde{\mathcal{G}}$, to be determined from the latter equality in (36). If the invariant scalar product $(\cdot, \cdot)_\delta$ on $\delta\tilde{\mathcal{G}}$ gives rise to a subalgebra splitting

$$\delta\tilde{\mathcal{G}} = \delta\tilde{\mathcal{G}}_+ \oplus \delta\tilde{\mathcal{G}}_-$$

with the analytical subalgebra $\delta\tilde{\mathcal{G}}_+$ isomorphic to the subalgebra $\tilde{\mathcal{G}}_+$, we can rewrite definition (36) as follows:

$$(\delta J_N(F, Q, R; \lambda), \delta X(\lambda))_\delta = H_{\delta X}(F, Q, R) = H_X(F, Q, F), \tag{37}$$

Corresponding calculations of the moment map $\delta J_N : M_{N,3}^3 \rightarrow \delta\tilde{\mathcal{G}}_+^*$ should give rise to the following important identity:

$$\delta J_N^+(F, Q, R; \lambda^*)|_{M_N} = S_N(f_j, g_j, p_j; \lambda) \tag{38}$$

for all $\lambda \in \mathbf{C}$. As a result of tedious but simple enough calculations the isomorphism $\delta : \tilde{\mathcal{G}} \rightleftharpoons \delta\tilde{\mathcal{G}}$ is built in the exact form which was not written here down because of its routine complexity. Now we have come in a position to formulate the following important theorem.

Theorem 3 *Let $\mathcal{I}(\delta\tilde{\mathcal{G}}_+^*)$ be the set of Casimir functionals on the adjoint space $\delta\mathcal{G}_+^*$. Then for each $\nu \in \mathcal{I}(\delta\tilde{\mathcal{G}}_+^*)$ the Hamiltonian flow with respect to the standard Lie-Poisson structure generates through the element $\delta J_N \in \delta\tilde{\mathcal{G}}_+^*$ on the matrix manifold $M_{N,3}^3$ a completely integrable via Liouville dynamical system equivalent to the following Lax type representation:*

$$\frac{d}{dt}(\delta J_N) = [grad \nu(\delta J_N)_+, \delta J_N]. \tag{39}$$

◁The proof of the above theorem is a simple consequence of the well-known Adler-Kostant-Symes theorem [7] on the complete integrability of flows on a direct sum splitted Lie coalgebra $\delta\tilde{\mathcal{G}}^*$, generated by usual coadjoint actions of the corresponding Lie group $\delta\tilde{\mathcal{G}}$ on it. ▷

To use the Theorem 3 in the case under consideration, we only need to describe the set of Casimir invariants $\mathcal{I}(\delta\tilde{\mathcal{G}}_+^*)$. The following lemma is true.

Lemma 3 *The set*

$$\mathcal{I}(\delta\tilde{\mathcal{G}}_+^*) = \left\{ \nu_k := \lambda^{kN_\lambda} \det(\lambda - A)^k Sp(\delta J_N)^k : k \in \mathbf{Z}_+ \right\}. \tag{40}$$

automatically satisfies the Casimir determining equation

$$[\text{grad } \nu_k(\delta J_N), (\delta J_N)] = 0 \tag{41}$$

for all $k \in \mathbf{Z}_+$, $\lambda \in \mathbf{C}$, on the manifold $M_{N,3}^3$.

For the case $k = 1$, we easily obtain that the following identification holds for all $\lambda \in \mathbf{C}$ upon the submanifold $M_N \subset M_{M,3}^3$:

$$L^+(u, v; \lambda^*) \Big|_{M_N} = L_N^+(f_j, g_j, p_j; \lambda^*) = (\lambda^{-N_\lambda} \det(\lambda - A) \cdot \delta J_N)_+ \Big|_{M_N}. \tag{42}$$

Hence from (38), (39), and (42), we can obtain the integrable flow generated by the Casimir functional $\nu_1 \in \mathcal{I}(\delta\tilde{\mathcal{G}}_+^*)$ for all $\lambda \in \mathbf{C}$:

$$dL_N/dx = [L_N, S_N]. \tag{43}$$

Expression (43) gives a usual Lax type representation for the Hamiltonian vector field d/dx on the invariant finite-dimensional submanifold $M_N \subset M$. This representation (43) completely coincides with that of (25) after its invariant reducing upon the submanifold $M_N \subset M$.

In addition, we can find also invariant functionally independent functions for the Hamiltonian flow d/dx on M_N , developing the independent Casimir functionals $\nu_k \in \mathcal{I}(\delta_+^*)$, $k = \overline{1, 3}$, in Laurent series as follows:

$$\nu_k \lambda^{kN_\lambda} \det(\lambda - A)^{-k} := \sum_{j=0}^{N_\lambda} \gamma_j^{(k)} / (\lambda - \lambda_j). \tag{44}$$

The set

$$\left\{ \gamma_j^{(k)} \in \mathcal{D}(M_N) : j = \overline{0, N_\lambda}, k = \overline{1, 3} \right\}$$

consists of $3(N_\lambda + 1)$ functionally independent conservation laws, involutive with respect to the symplectic structure (22), for the Hamiltonian vector field d/dx on M_N . Thereby the following concluding theorem is true.

Theorem 4 *The Lax type dynamical system (43) on the coadjoint space δG_+^* is equivalent to the set of eigenfunction equations (8), (9) and generates on the manifold $M_N \subset M$ the completely integrable Hamiltonian vector field d/dx with respect to the noncanonical symplectic structure (22) and the Hamiltonian function (23).*

Using further the algebro-geometric considerations of the 2π -periodic spectral problem (8) on the real axis \mathbf{R} , one can successfully find the corresponding eigenfunctions in exact form, thereby one can find an exact set of solutions to the initially given hydrodynamic Boussinesq equation (1) on the nonlocal finite-dimensional invariant submanifold $M_N \subset M$. The procedure described above gives rise also to the second, both efficient and nontrivial way for obtaining finite-dimensional reductions of Lax type integrable nonlinear dynamical systems in question.

3 Neumann-type oscillatory super-Hamiltonian systems on the sphere \mathbf{S}^N and their Lie algebraic superintegrability

3.1 The super-Hamiltonian Korteweg-de Vries dynamical system (4) appears most naturally as an orbit on the coadjoint space $\widehat{\mathcal{G}}^*$ to the affine super-conformal Lie algebra $\widehat{\mathcal{G}}$ of central extended super-conformal vector fields on the super-circle $\mathbf{S}^{1|1}$, as was firstly shown in [8]. Let us consider the affine spanning $\widetilde{\mathcal{G}}$ of super-conformal vector fields on the super-circle $\mathbf{S}^{1|1}$:

$$\widetilde{\mathcal{G}} := \mathcal{G} \otimes \mathbf{C}(\lambda, \lambda^{-1}),$$

where $\lambda \in \mathbf{C}$ and the super-conformal Lie algebra \mathcal{G} consists of the following vector fields on $\mathbf{S}^{1|1}$:

$$\mathcal{G} := \left\{ \mathcal{F} \frac{\partial}{\partial x} + \frac{1}{2} (D_\theta \mathcal{F}) D_\theta : \mathcal{F} = f_0 + \theta f_1 \in \Lambda := \Lambda_0 \oplus \Lambda_1 \right\}, \tag{45}$$

where $\Lambda = \Lambda_0 \oplus \Lambda_1$ is a usual scalar Grassmann algebra, $D_\theta = \partial/\partial\theta + \theta\partial/\partial x$. The central extension $\widehat{\mathcal{G}}$ of the Lie algebra \mathcal{G} is given by means of the real meaningful two-cocycle $c(\mathcal{F}, \mathcal{Q})$ on $\widehat{\mathcal{G}}$,

$$c(\mathcal{F}, \mathcal{Q}) := \text{res}_{\lambda=\infty} \int_0^{2\pi} dx \int d\theta (\mathcal{F} D_\theta^5 \mathcal{Q}) = \text{res}_{\lambda=\infty} \int_0^{2\pi} dx (f g_{3x} + \alpha \beta_{2x}), \tag{46}$$

where by definition, $\mathcal{F} := f + \theta\alpha$, $\mathcal{Q} := g + \theta\beta \in \Lambda^{1|1}$. The corresponding commutator of elements (\mathcal{F}, c_1) and $(\mathcal{Q}, c_2) \in \widehat{\mathcal{G}}$ has the form :

$$[(\mathcal{F}, c_1), (\mathcal{Q}, c_2)] = ([\mathcal{F}, \mathcal{Q}], c(\mathcal{F}, \mathcal{Q}) + c_0(\mathcal{F}, \mathcal{Q})), \tag{47}$$

where $c_1, c_2 \in \mathbf{R}$, $c_0(\mathcal{F}, \mathcal{Q}) := \text{res}_{\lambda=\infty} \int_0^{2\pi} dx (f g_x + \frac{1}{4} \alpha \beta)$ is a trivial 2-cocycle on $\widehat{\mathcal{G}}$ added for convenience. The Lie super-algebra $\widehat{\mathcal{G}}$ as a set of Laurent series allows in $\lambda \in \mathbf{C}$ the standard direct sum splitting into two subalgebras:

$$\widehat{\mathcal{G}} = \widehat{\mathcal{G}}_+ \oplus \widehat{\mathcal{G}}_-.$$

The dual space $\widetilde{\mathcal{G}}^*$ consists of formal series of the following kind:

$$w(x, \theta; \lambda) = \sum_{j \in \mathbf{Z}} w_j(x, \theta) \lambda^j$$

where $w_j(x, \theta) \in \mathcal{C}^\infty(\mathbf{S}^{1|1}; \mathbf{R}^{1|1})$, $j \in \mathbf{Z}$.

Linear functionals on $\widehat{\mathcal{G}}$ are defined as follows:

$$((w, c_1), (\mathcal{F}, c_2)) := c_1 c_2 + \text{res}_{\lambda=\infty} \int_0^{2\pi} dx (w \mathcal{F}), \tag{48}$$

where $c_1, c_2 \in \mathbf{R}$ are arbitrary real values.

It is easy to see that the group $\widetilde{\mathcal{G}}_-$ of super-conformal transformations of the super-circle $\mathbf{S}^{1|1}$ acts on the dual space $\widehat{\mathcal{G}}_-^* \cong \widehat{\mathcal{G}}_+$ as follows:

$$\text{Ad}_\varphi^*(w, c) = \left((D_{\widetilde{\theta}}^3 w) \circ \varphi^{-1} - (D_{\widetilde{\theta}}^3 \theta)^{-2} [(D_{\widetilde{\theta}}^4 \theta)(D_{\widetilde{\theta}}^4 \theta) - 2(D_{\widetilde{\theta}}^3 \theta)(D_{\widetilde{\theta}}^2 \theta)], c \right), \tag{49}$$

where $\mathbf{S}^{1|1} \ni (x, \theta) \xrightarrow{\varphi} (\widetilde{x}, \widetilde{\theta}) \in \mathbf{S}^{1|1}$ is an arbitrary transformation, $D_{\widetilde{\theta}} := \partial/\partial\widetilde{\theta} + \widetilde{\theta}\partial/\partial\widetilde{x}$ by definition.

The next lemma explains the nature of the Lax representation (5) from the above considerations.

Lemma 4 *The infinitesimal version of (49) gives rise to a full description of the whole infinite hierarchy of Lax type superintegrable higher Korteweg-de Vries equations with the Lax operator given in the form (5).*

◁ Let $\varphi_t \in \text{Diff } \mathbf{S}^{1|1}$, $t \in \mathbf{R}$, is any one- parametric super-conformal affine transformation of the super-circle $\mathbf{S}^{1|1}$. Its action on an element $(w, c) \in \widehat{\mathcal{G}}_-^*$, given by (49), generates a vector field on $\widehat{\mathcal{G}}_-^*$ as follows:

$$dw/dt = \frac{d}{dt} Ad_{\varphi_t}^*(w, c) = \left[-D_\theta^3 + w, P(w) \right] \tag{50}$$

where, by definition, $d\varphi_t/dt = \mathcal{F} \in \widetilde{\mathcal{G}}_-$ is a super-conformal vector field on $\mathbf{S}^{1|1}$ generated by some Casimir functional $\gamma \in \mathcal{I}(\widetilde{\mathcal{G}}^*)$ reduced properly upon the subspace $\widehat{\mathcal{G}}_-^*$, $P(w)$ is the correspondingly calculated super-differential operator acting in the functional space $\mathcal{W}_\infty^{(3)}(\mathbf{S}^{1|1}; \mathbf{C}^{1|1})$. Putting in (50) the element $w := (u - \lambda)\theta + \xi \in \widetilde{\mathcal{G}}_-^* \cong \widetilde{\mathcal{G}}_+^*$, we can easily find the spectral problem (5) associated with the Lax type representation (50). The latter proves the lemma. ▷

3.2 Using the spectral problem (5), we can easily find the following local conservation laws of the super-Korteweg-de Vries dynamical system (4):

$$\begin{aligned} \gamma_0 &= \int_0^{2\pi} u \, dx, & \gamma_1 &= \int_0^{2\pi} dx (u^2 - 4\xi\xi_x), \\ \gamma_2 &= \int_0^{2\pi} dx (u_x^2 + 2u^3 - 16\xi_x\xi_{2x} - 24u\xi\xi_x), & \dots \end{aligned} \tag{51}$$

and so on. The invariant functionals are in involution with respect to the following Lie-Poisson structure on the space $\widehat{\mathcal{G}}_-^*$:

$$\{\gamma, \mu\}_0(w) := (w, [\text{grad } \gamma_-(w), \text{grad } \mu_-(w)]) \tag{52}$$

for any $w \in \widehat{\mathcal{G}}_-^* \cong \widehat{\mathcal{G}}_+^*$. At the element $w = (u - \lambda)\theta + \xi \in \widehat{\mathcal{G}}_-^*$, the bracket (52) gives rise to the following Poisson brackets on the super-manifold $M^{1|1} \ni (u, \xi)^\tau$, found before in [8, 9]:

$$\begin{aligned} \{u(x), u(y)\} &= -\frac{1}{2}\delta_{3x}(x - y) + 2u(x)\delta_x(x - y) + u_x\delta(x - y), \\ \{u(x), \xi(y)\} &= \frac{3}{2}\xi(x)\delta_x(x - y) + \frac{1}{2}\xi_x(x)\delta(x - y), \\ \{\xi(x), \xi(y)\} &= \frac{1}{2} \left(u(x) - \frac{d^2}{dx^2} \right) \delta(x - y), \end{aligned} \tag{53}$$

for all $x, y \in \mathbf{S}^1$. Thus, the dynamical system (4) on the supermanifold $M^{1|1}$ is a super-Hamiltonian one having the following representation:

$$du/dt = \{\gamma_1, u\}, \quad d\xi/dt = \{\gamma_1, \xi\}. \tag{54}$$

The super-Hamiltonian system (54) has also a countable hierarchy of invariant finite-dimensional submanifolds $M_N^{1|1} \subset M^{1|1}$ for all $N \in \mathbf{Z}_+$, upon which the vector fields d/dt and d/dx are canonically super-Hamiltonian Liouville integrable Neumann type oscillatory systems.

To proceed to this topic in detail, let us build the following invariant finite-dimensional super-submanifold:

$$M_N^{1|1} := \left\{ (u, \xi)^\tau \in M^{1|1} : \text{grad } \mathcal{L}_N[u, \xi] = 0 \right\}, \tag{55}$$

where, by definition,

$$\mathcal{L}_N := -\gamma_{N_\gamma+1} + \sum_{j=0}^{N_\gamma} a_j \gamma_j + \sum_{j=0}^{N_\lambda} b_j \lambda_j, \tag{56}$$

$\gamma_j \in \mathcal{D}(M^{1|1})$, $j = \overline{0, N_\gamma}$ are local conservation laws for (4), $\lambda_j \in \mathcal{D}(M^{1|1})$, $j = \overline{0, N_\lambda}$, are the corresponding nonlocal real conservation laws for (4) being equal to conserved eigenvalues of the periodic spectral problem (5), $a_j \in \mathbf{R}$, $j = \overline{0, N_\gamma}$, $b_k \in \mathbf{R}$, $k = \overline{0, N_\lambda}$, are fixed but arbitrary real numbers, $N_\lambda, N_\gamma \in \mathbf{Z}_+$. In the case where $N_\gamma + 1 = 0$, $b_j = \int_0^{2\pi} f_j^2 dx$, $j = \overline{0, N_\lambda}$, the super-manifold (55) takes the form:

$$M_N^{1|1} = \left\{ (u, \xi)^\tau \in M^{1|1} : \sum_{j=0}^{N_\lambda} q_j^2 = 1, \sum_{j=0}^{N_\lambda} q_j p_j = 0, \sum_{j=0}^{N_\lambda} q_j \alpha_j = 0 \right\}, \tag{57}$$

where we have put, by definition, $f_j := q_j + \theta \alpha_j \in \mathcal{W}_\infty^{(3)}(\mathbf{S}^{1|1}; \mathbf{R}^{1|1})$, $p_j := dq_j/dx$, $j = \overline{0, N_\lambda}$. For the reduction theory [5, 6] to be used effectively one needs to build a extended super-submanifold $M_N^{1|1}$ as follows:

$$\begin{aligned} \overline{M}_N^{1|1} \cong & \left\{ (u, \xi; q_j, p_j, \alpha_j)^\tau \in M^{1|1} \otimes \mathcal{W}_\infty^{(3)} \otimes \mathcal{W}_\infty^{(2)} \otimes \mathcal{W}_\infty^{(1)} : \right. \\ & \left. -D_\theta^3 f_j + [u\theta + \xi]f_j = \lambda_j \xi f_j, \quad f_j = q_j + \theta \alpha_j, \quad j = \overline{0, N_\lambda} \right\}. \end{aligned} \tag{58}$$

It is easy to see that the super-submanifold $M_N^{(1/1)} \subset M^{1|1}$ in jet-coordinates $\{(q_j, p_j, \alpha_j)^\tau \in \mathbf{R}^{2|1}, j = \overline{0, N_\lambda}\}$ is isomorphic to some product of the coadjoint space $T^*(\mathbf{S}^{N_\lambda})$ to the sphere \mathbf{S}^{N_λ} and projective $(N_\lambda - 1)$ -dimensional hypersurface $H^{N_\lambda} \subset P_{N_\lambda}(\mathbf{R}^{0|1})$.

To formulate further the theorem on integrability of the resulting Neumann type oscillatory super-dynamical system on the super-manifold $M_N^{1|1}$, we need the following

Lemma 5 *The super-manifold $M_N^{1|1} \cong T^*(\mathbf{S}^{N_\lambda}) \otimes H^{N_\lambda-1}$ carries the following canonically built super-symplectic structure*

$$\omega^{(2)} = \sum_{j=0}^{N_\lambda} (dp_j \wedge dq_j + d\alpha_j \wedge d\alpha_j) |_{T^*(\mathbf{S}^{N_\lambda}) \otimes H^{N_\lambda-1}}, \tag{59}$$

with respect to which the vector field d/dx on $M_N^{1|1}$ is a super-Hamiltonian one with a Hamiltonian function $h^{(x)} \in \mathcal{D}(M_N^{1|1})$, where

$$h^{(x)} = \frac{1}{2} \sum_{j=0}^{N_\lambda} (\omega_j q_j^2 + p_j^2), \tag{60}$$

$\lambda_j := \omega_j \neq \omega_k \in \mathbf{R}$ at all $j \neq k = \overline{0, N_\lambda}$.

◁ The proof of the lemma is stemming from the following main determining relationship for the differential $d\mathcal{L}_N[u, \xi]$:

$$d\mathcal{L}_N[u, \xi] = \langle \text{grad } \mathcal{L}_N[u, \xi], (du, d\xi)^\tau \rangle + d\beta^{(1)}[u, \xi]/dx, \tag{61}$$

where $\beta^{(1)} = \sum_{j=0}^{N_\lambda} (p_j dq_j - q_j dp_j + 2\alpha_j d\alpha_j) |_{M_N^{1|1}}$. Using the definition $\omega^{(2)} := 1/2d\beta^{(1)}$, we immediately obtain (59). Analogously from the defining expression

$$\langle \text{grad } \mathcal{L}_N[u, \xi], (u_x, \xi_x)^\tau \rangle = -dh^{(x)}/dx, \tag{62}$$

one properly finds that a function $h^{(x)} \in \mathcal{D}(M_N^{1|1})$ coincides exactly with that of (60). The lemma is proved. ▷

Lemma 6 *The vector field d/dx on the extended super-submanifold $\overline{M}_N^{1|1}$ is a super-Hamiltonian one with respect to the canonical symplectic structure*

$$\overline{\omega}^{(2)} = \sum_{j=0}^{N_\lambda} (dp_j \wedge dq_j + d\alpha_j \wedge d\alpha_j) \tag{63}$$

and the Hamiltonian function

$$\overline{h}^{(x)} = \frac{1}{2} \sum_{j=0}^{N_\lambda} (\omega_j q_j^2 + p_j^2) + u \left(\sum_{j=0}^{N_\lambda} q_j^2 - 1 \right) + 2\xi \sum_{j=0}^{N_\lambda} q_j \alpha_j, \tag{64}$$

where

$$u = \sum_{j=0}^{N_\lambda} (\omega_j q_j^2 - p_j^2), \quad \xi = - \sum_{j=0}^{N_\lambda} p_j \alpha_j. \tag{65}$$

The expressions $\sum_{j=0}^{N_\lambda} q_j^2 - 1 = 0$, $\sum_{j=0}^{N_\lambda} q_j p_j = 0$, and $\sum_{j=0}^{N_\lambda} q_j \alpha_j = 0$ are conserved quantities of the vector field d/dx on $M_N^{1|1}$:

$$dq_j/dx = p_j, \quad dp_j/dx = \omega_j q_j + u q_j - \xi \alpha_j, \quad d\alpha_j/dx = \xi q_j \tag{66}$$

for all $j = \overline{0, N_\lambda}$.

Using conditions (65), constraints $\sum_{j=0}^{N_\lambda} q_j^2 - 1 = 0$, $\sum_{j=0}^{N_\lambda} q_j p_j = 0$ and $\sum_{j=0}^{N_\lambda} q_j \alpha_j = 0$, the proper Dirac reduction [2, 10] of the dynamical system (66) upon the invariant super-submanifold $M_N^{1|1} \subset \overline{M}_N^{1|1}$ yields the above stated results of Lemma 4 on the invariant super-submanifold $M_N^{1|1} \cong T^*(\mathbf{S}^N) \otimes H^{N_\lambda-1}$.

Now we are in a position to formulate the following theorem.

Theorem 5 *The vector field d/dx on the invariant super-submanifold $M_N^{1|1}$ is a Liouville integrable super-Hamiltonian system equivalent to the flow (66), having a Lax type representation*

$$dJ_N/dx = [L_N, J_N], \tag{67}$$

where, by definition, $L_N := L|_{M_N^{1|1}}$ via the Moser mapping,

$$L = \begin{pmatrix} 0 & 1 & 0 \\ u - \lambda & 0 & -\xi \\ \xi & 0 & 0 \end{pmatrix},$$

$$J_N = \sum_{j=0}^{N_\lambda} \frac{1}{\lambda - \omega_j} \begin{pmatrix} -q_j p_j & -p_j^2 & p_j \alpha_j \\ q_j^2 & q_j p_j & -q_j \alpha_j \\ -q_j \alpha_j & -p_j \alpha_j & 0 \end{pmatrix} + \sum_{j=0}^{N_\lambda} \frac{\xi}{\lambda - \omega_j} \begin{pmatrix} 0 & q_j \alpha_j & 0 \\ 0 & 0 & 2q_j^2 \\ 0 & -2q_j^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{68}$$

The matrix mapping $J_N : M_N^{1|1} \rightarrow \widetilde{su}^*(2|1)$ is a corresponding moment map generated by the following natural group action of the affine Lie super-group $\widetilde{SU}(2|1)$ on the manifold $M_N^{1|1}$, imbedded in the symplectic matrix super-space $M_{N,2|1} \times M_{N,2|1}$ via the reduction $Q = F^* \sigma$,

$$\sigma := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the canonical symplectic structure

$$\Omega^{(2)} := -Sp(dF \wedge dQ^\tau) \tag{69}$$

similarly to that constructed in (27):

$$F \rightarrow F_g := \text{res}_{\lambda=\infty} \left(\frac{1}{\lambda - A} Fg(\lambda) \right),$$

$$Q^\tau \rightarrow Q_g^\tau := \text{res}_{\lambda=\infty} \left(g(\lambda) Q^\tau \frac{1}{\lambda - A} \right), \tag{70}$$

where $g(\lambda) \in \widetilde{SU}(2|1)$, $\lambda \in \mathbf{C}$ is the affine parameter and the matrix

$$A := \text{diag} \left\{ \omega_j \neq \omega_k \in \mathbf{R}_+ \text{ at } j \neq k = \overline{0, N_\lambda} \right\}.$$

◁ The proof of the theorem above is completely similar to that of chapter 2 in this paper, on what we will not stay here in detail. Note only that a complete system of involutive conservation laws is given by the following simple expressions on the matrix super-space $M_{N,2|1}$:

$$\nu_k = \text{sdet}(\lambda - A)^k \lambda^{-kN_\lambda} {}_sSp J_N^k, \tag{71}$$

$k = \overline{1, 3}$. Using an expansion similar to (44), we can find three finite hierarchies of conserved functions in involution on the supermanifold $M_N^{1|1} \cong T^*(\mathbf{S}^N) \otimes H^{N_\lambda-1}$ supplying the wanted conditions for the Liouville integrability theorem to be used successfully. The latter proves the theorem. ▷

The results obtained above in chapter 3 for the oscillatory Neumann type super-Hamiltonian system (66) on $T^*(\mathbf{S}^{N_\lambda}) \otimes H^{N_\lambda-1}$ can be easily generalized to the integrability theorem for the oscillatory Neumann-Rosokhatius super-Hamiltonian system on the supermanifold $T^*(\mathbf{S}^{N_\lambda}) \otimes H^{N_\lambda-1}$ via the well-known Marsden-Weinstein reduction procedure for some special Lie super-group action on it. These and some other results will be presented in detail in a sequel paper under preparation.

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