

# How to Find Discrete Contact Symmetries

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## Abstract

This paper describes a new algorithm for determining all discrete contact symmetries of any differential equation whose Lie contact symmetries are known. The method is constructive and is easy to use. It is based upon the observation that the adjoint action of any contact symmetry is an automorphism of the Lie algebra of generators of Lie contact symmetries. Consequently, all contact symmetries satisfy various compatibility conditions. These conditions enable the discrete symmetries to be found systematically, with little effort.

## 1. Introduction

Discrete symmetries of differential equations are used in various ways. They map solutions to (possibly new) solutions. They may be used to create efficient numerical methods for the computation of solutions to boundary-value problems. Indeed, there is currently much research into techniques for constructing numerical methods that preserve various types of symmetry [2, 4, 10]. Discrete and continuous groups of symmetries determine the nature of bifurcations in nonlinear dynamical systems. Equivariant bifurcation theory describes the effects of symmetries, but it may yield misleading results unless *all* symmetries of the dynamical system are known [3, 6].

In general, it is straightforward to find all one-parameter Lie groups of symmetries of a given system, using techniques developed by Sophus Lie more than a century ago [1, 11, 12, 14]. Yet, until recently, no simple method for finding all discrete symmetries was known. Ansatz-based methods can be used to find discrete symmetries belonging to particular classes, e.g. [5], but such methods cannot guarantee that all discrete symmetries have been found. The main difficulty is that, commonly, the determining equations for discrete symmetries form a highly-coupled nonlinear system. Reid and co-workers have developed a computer algebra package aimed at reducing this system to a differential Gröbner basis [13], but the method is computationally intensive and seems not to have been widely used.

A new approach to the problem of finding discrete point symmetries has recently been described by the author [7]. Instead of trying to solve the symmetry condition directly, one first examines the adjoint action of an arbitrary discrete point symmetry upon the Lie algebra of Lie point symmetry generators. This yields a set of necessary conditions which simplify the problem of constructing all discrete point symmetries. Of course, one must know the Lie algebra of Lie point symmetry generators, which must be non-trivial. However, this is not a severe limitation, as most differential equations of physical importance have a non-trivial Lie algebra.

Most techniques of Lie symmetry analysis readily generalize to contact symmetries, and the aim of the current paper is to show how all discrete contact symmetries of a given differential equation may be found systematically, using an extension of the algorithm for determining discrete point symmetries. Perhaps surprisingly, many differential equations without non-point Lie contact symmetries have non-point discrete contact symmetries. It is useful to be able to derive these (non-obvious) symmetries systematically because, like all symmetries, they constrain the behaviour of solutions of the differential equation.

The new method is described first in the context of ordinary differential equations (ODEs) of order  $n \geq 3$ , then adapted to treat a finite subgroup of the infinite group of contact symmetries of a given second-order ODE. Finally, it is shown that the method can be extended to partial differential equations (PDEs).

## 2. The algorithm applied to ODEs

A diffeomorphism

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$$

is a symmetry of the ODE

$$y^{(n)} = \omega(x, y, y', \dots, y^{(n-1)}) \quad (2.1)$$

if it maps the set of solutions to itself, i.e. if

$$\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \dots, \hat{y}^{(n-1)}) \quad \text{when (2.1) holds.} \quad (2.2)$$

Here the functions  $\hat{y}^{(k)}$  are obtained by prolonging the diffeomorphism  $\Gamma$  to derivatives, using

$$\hat{y}^{(k)} \equiv \frac{D\hat{y}^{(k-1)}}{D\hat{x}}, \quad (\hat{y}^{(0)} = \hat{y}), \quad (2.3)$$

where

$$D = \partial_x + y'\partial_y + y''\partial_{y'} + \dots$$

For point symmetries,  $\hat{x}$  and  $\hat{y}$  are functions of  $x$  and  $y$ . Contact symmetries are more general than point symmetries, because  $\hat{x}$ ,  $\hat{y}$  and  $\hat{y}'$  are functions of  $x$ ,  $y$  and  $y'$ . One-parameter Lie groups of contact symmetries are obtained in a similar way to Lie point symmetries, by linearizing the symmetry condition (2.2). Specifically,

$$\begin{aligned} \hat{x} &= x + \epsilon\xi(x, y, y') + O(\epsilon^2), \\ \hat{y} &= y + \epsilon\eta(x, y, y') + O(\epsilon^2), \\ \hat{y}^{(k)} &= y^{(k)} + \epsilon\eta^{(k)}(x, y, y', \dots, y^{(k)}) + O(\epsilon^2), \quad k \geq 1. \end{aligned}$$

The contact conditions require that the  $O(\epsilon)$  terms should be expressible in terms of the characteristic function  $Q(x, y, y') = \eta - y'\xi$ , as follows

$$\xi = -Q_{y'}, \quad \eta = Q - y'Q_{y'}, \quad \eta^{(k)} = D^k Q - y^{(k+1)}Q_{y'}, \quad k \geq 1. \quad (2.4)$$

In particular,

$$\eta^{(1)}(x, y, y') = Q_x + y'Q_y.$$

Lie point symmetries of ODEs of order  $n \geq 2$  are Lie contact symmetries whose characteristic function is linear in  $y'$ .

The set of all infinitesimal generators of Lie contact symmetries of a given ODE of order  $n \geq 3$  forms a finite-dimensional Lie algebra,  $\mathcal{L}$ , which can generally be determined systematically [14]. Given such an ODE, suppose that  $\mathcal{L}$  has a basis

$$X_i = \xi_i(x, y, y')\partial_x + \eta_i(x, y, y')\partial_y + \eta_i^{(1)}(x, y, y')\partial_{y'}, \quad i = 1, \dots, N, \quad (2.5)$$

where  $(\xi_i, \eta_i, \eta_i^{(1)})$  are obtained from the characteristic function  $Q_i(x, y, y')$  using (2.4), and  $N = \dim(\mathcal{L})$ . The structure constants,  $c_{ij}^k$ , for the basis (2.5) are determined by

$$[X_i, X_j] = c_{ij}^k X_k. \quad (2.6)$$

(Summation is implied when an index occurs twice, one raised and once lowered.) The one-parameter Lie group of contact symmetries corresponding to a particular  $X_i$  is obtained by exponentiation. We use the notation

$$\Gamma_i(\epsilon) : (x, y, y') \mapsto (e^{\epsilon X_i} x, e^{\epsilon X_i} y, e^{\epsilon X_i} y').$$

Suppose that

$$\Gamma : (x, y, y') \mapsto (\hat{x}(x, y, y'), \hat{y}(x, y, y'), \hat{y}'(x, y, y')) \quad (2.7)$$

is a contact symmetry of the given ODE. Then the contact transformation obtained by the adjoint action of  $\Gamma$  upon  $\Gamma_i(\epsilon)$ ,

$$\hat{\Gamma}_i(\epsilon) = \Gamma \Gamma_i(\epsilon) \Gamma^{-1}, \quad (2.8)$$

is also a contact symmetry, for each  $\epsilon$  in some neighbourhood of zero. Therefore, for each  $i$ , there is a (local) one-parameter Lie group of contact symmetries

$$\hat{\Gamma}_i(\epsilon) : (\hat{x}, \hat{y}, \hat{y}') \mapsto (e^{\epsilon \hat{X}_i} \hat{x}, e^{\epsilon \hat{X}_i} \hat{y}, e^{\epsilon \hat{X}_i} \hat{y}'), \quad (2.9)$$

whose infinitesimal generator is

$$\hat{X}_i = \Gamma X_i \Gamma^{-1} = \xi_i(\hat{x}, \hat{y}, \hat{y}')\partial_{\hat{x}} + \eta_i(\hat{x}, \hat{y}, \hat{y}')\partial_{\hat{y}} + \eta_i^{(1)}(\hat{x}, \hat{y}, \hat{y}')\partial_{\hat{y}'}. \quad (2.10)$$

Consequently

$$\hat{X}_i \in \mathcal{L}, \quad i = 1, \dots, N.$$

The generators  $\{\hat{X}_i\}_{i=1}^N$  are simply the basis generators  $\{X_i\}_{i=1}^N$  with  $(x, y, y')$  replaced by  $(\hat{x}, \hat{y}, \hat{y}')$ . Therefore the set  $\{\hat{X}_i\}_{i=1}^N$  is a basis for  $\mathcal{L}$ , and so each  $X_i$  can be written as

a linear combination of the  $\hat{X}_j$ 's. Also, the mapping  $X_i \mapsto \hat{X}_i$  is an automorphism of  $\mathcal{L}$  which preserves all structure constants, i.e.

$$[\hat{X}_i, \hat{X}_j] = c_{ij}^k \hat{X}_k \quad \text{when (2.6) holds.} \quad (2.11)$$

These results generalize to partial differential equations, and are summarized as follows.

**Lemma 1.** *Every contact symmetry  $\Gamma$  of an ordinary differential equation of order  $n \geq 3$  induces an automorphism of the Lie algebra,  $\mathcal{L}$ , of generators of one-parameter local Lie groups of contact symmetries of the differential equation. For each such  $\Gamma$ , there exists a constant non-singular matrix  $(b_i^l)$  such that*

$$X_i = b_i^l \hat{X}_l. \quad (2.12)$$

*This automorphism preserves all structure constants.*

Lemma 1 yields the following PDEs for the unknown functions  $\hat{x}(x, y, y')$  and  $\hat{y}(x, y, y')$ :

$$X_i \hat{x} = b_i^l \hat{X}_l \hat{x} = b_i^l \xi_l(\hat{x}, \hat{y}, \hat{y}'), \quad i = 1, \dots, N, \quad (2.13)$$

$$X_i \hat{y} = b_i^l \hat{X}_l \hat{y} = b_i^l \eta_l(\hat{x}, \hat{y}, \hat{y}'), \quad i = 1, \dots, N. \quad (2.14)$$

This set of  $2N$  first-order PDEs, together with the contact condition

$$\hat{y}'(x, y, y') = \frac{\frac{d\hat{y}}{dx}}{\frac{d\hat{x}}{dx}},$$

provides necessary, but not sufficient, conditions for  $\Gamma$  to be a contact symmetry. The contact condition yields the following pair of PDEs, because  $\hat{y}'$  is independent of  $y''$ :

$$\hat{y}_x + y' \hat{y}_y = (\hat{x}_x + y' \hat{x}_y) \hat{y}', \quad \hat{y}_{y'} = \hat{x}_{y'} \hat{y}'. \quad (2.15)$$

N.B. The lemma gives a further  $N$  PDEs

$$X_i \hat{y}' = b_i^l \hat{X}_l \hat{y}' = b_i^l \eta_l^{(1)}(\hat{x}, \hat{y}, \hat{y}'), \quad i = 1, \dots, N,$$

but these add nothing new, for they are a consequence of (2.13), (2.14) and the contact condition.

To find all discrete contact symmetries, proceed as follows. First solve the system of PDEs (2.13), (2.14), to obtain  $(\hat{x}, \hat{y})$  in terms of  $x, y, y', b_i^l$  and some unknown constants (or functions) of integration. If  $N$  is sufficiently large, it may be possible to solve this system algebraically (by eliminating the derivative terms); otherwise, the method of characteristics should be used. Incorporate the contact condition (2.15); this generally reduces the number of candidate solutions  $(\hat{x}, \hat{y})$ . Finally, use the symmetry condition (2.2) to determine which of these solutions are symmetries. The continuous symmetries may be factored out at a convenient point in the calculation. The remaining discrete symmetries are inequivalent under any continuous symmetry, and form a discrete (but not necessarily finite) group.

If  $\mathcal{L}$  is non-abelian, some of the structure constants are non-zero, enabling the matrix  $B = (b_i^l)$  to be simplified *before* any of the above calculations are done. Substituting (2.12)

into (2.6), and taking (2.11) into account, we obtain the following constraints on the components of  $B$ :

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n. \quad (2.16)$$

It is sufficient to restrict attention to equations (2.16) with  $i < j$ , because the structure constants are antisymmetric in the two lower indices. Moreover, at least some of the continuous symmetries can be factored out using their adjoint action upon the generators in  $\mathcal{L}$ , (see [11]),

$$\text{Ad}(\exp(\epsilon_j X_j)) X_i = X_i - \epsilon_j [X_j, X_i] + \frac{\epsilon_j^2}{2!} [X_j, [X_j, X_i]] - \dots = a_i^p(\epsilon_j, j) X_p. \quad (2.17)$$

Let  $A(j)$  denote the matrix whose components are  $a_i^p(\epsilon_j, j)$ , as defined by (2.17). The system (2.12) is equivalent, under the group generated by  $X_j$ , to

$$X_i = \tilde{b}_i^l \hat{X}_l,$$

where  $\tilde{b}_i^l$  are the components of

$$\tilde{B} = A(j)B. \quad (2.18)$$

The mapping  $B \mapsto \tilde{B}$  does not affect (2.16), so we will drop tildes as soon as each equivalence transformation has been made. Each generator  $X_j$  is used in turn to simplify the form of  $B$ .

To illustrate this procedure, consider the two-dimensional non-abelian Lie algebra  $\mathfrak{a}(1)$ , with a basis  $\{X_1, X_2\}$  such that

$$[X_1, X_2] = X_1. \quad (2.19)$$

The only non-zero structure constants are

$$c_{12}^1 = -c_{21}^1 = 1.$$

Therefore (2.16) gives

$$b_1^1 b_2^2 - b_1^2 b_2^1 = b_1^1, \quad 0 = b_1^2,$$

and hence

$$B = \begin{bmatrix} b_1^1 & 0 \\ b_2^1 & 1 \end{bmatrix}, \quad b_1^1 \neq 0.$$

The matrices representing the adjoint action of the continuous group on the Lie algebra are

$$A(1) = \begin{bmatrix} 1 & 0 \\ -\epsilon_1 & 1 \end{bmatrix}, \quad A(2) = \begin{bmatrix} e^{\epsilon_2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying (2.18), first with  $j = 1$ ,  $\epsilon_1 = \frac{b_2^1}{b_1^1}$ , then with  $j = 2$ ,  $\epsilon_2 = -\ln |b_1^1|$ , we obtain

$$B = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{where } \alpha \in \{-1, 1\}. \quad (2.20)$$

The reduced form of the matrix  $B$  is specific to this particular Lie algebra, and is independent of the ODE whose Lie point symmetries are generated by the algebra.

Simplified matrices for other non-abelian Lie algebras can be found by the same technique. However, if  $\mathcal{L}$  is abelian, the entries of  $B$  cannot be determined *a priori*.

### 3. Examples

The third-order ODE

$$y''' = \frac{y''^2}{x} - \frac{y''}{y'} \quad (3.1)$$

has a two-dimensional Lie algebra of generators of Lie contact symmetries. These Lie symmetries are actually point symmetries, and  $\mathcal{L}$  is isomorphic to  $\mathfrak{a}(1)$ . The basis of  $\mathcal{L}$ ,

$$X_1 = \partial_y, \quad X_2 = \frac{x}{2}\partial_x + y\partial_y + \frac{y'}{2}\partial_{y'},$$

has the commutation relations (2.19), and therefore  $B$  is given by (2.20). The system of PDEs (2.13), (2.14) amounts to

$$\begin{bmatrix} X_1\hat{x} & X_1\hat{y} \\ X_2\hat{x} & X_2\hat{y} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{\hat{x}}{2} & \hat{y} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \frac{\hat{x}}{2} & \hat{y} \end{bmatrix}.$$

The general solution of this system is

$$\hat{x} = xp(t), \quad \hat{y} = \alpha y + x^2q(t), \quad \text{where } t = \frac{y'}{x}.$$

The contact condition gives

$$\hat{y}' = xr(t),$$

where

$$\alpha t + 2q - t\dot{q} = (p - t\dot{p})r \quad \text{and} \quad \dot{q} = \dot{p}r.$$

(Here a dot over a function denotes its derivative with respect to  $t$ .) Re-arranging these conditions, we obtain

$$q = \frac{1}{2}(pr - \alpha t), \quad (3.2)$$

and the contact condition is satisfied if and only if

$$p\dot{r} - \dot{p}r = \alpha. \quad (3.3)$$

It is convenient to work in terms of  $p$  and  $r$ , because the ODE is invariant under translations in  $y$ , and so  $y$  does not occur in the symmetry condition. The prolongation to second and third derivatives is

$$\hat{y}'' = \frac{r + (y'' - t)\dot{r}}{p + (y'' - t)\dot{p}},$$

$$\hat{y}''' = \frac{(p\dot{r} - \dot{p}r)xy'''' + (y'' - t)^2(p\ddot{r} - \dot{p}\dot{r}) + (y'' - t)^3(\dot{p}\ddot{r} - \ddot{p}\dot{r})}{x(p + (y'' - t)\dot{p})^3}.$$

These expressions are substituted into the symmetry condition (2.2), and powers of  $y''$  are equated to yield an over-determined system of nonlinear ODEs for  $p$  and  $r$ . These are

easily solved with the aid of the contact condition (3.3). There are two sets of solutions. Either

$$(p, r) = (c, ct), \quad c^2 = \alpha, \quad (3.4)$$

or

$$(p, r) = (ct, c), \quad c^2 = -\alpha. \quad (3.5)$$

Re-writing these solutions in terms of the original variables, we obtain eight inequivalent discrete symmetries that form a group isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$ . The group generators are

$$\Gamma_1 : (x, y, y') \mapsto (ix, -y, iy'),$$

$$\Gamma_2 : (x, y, y') \mapsto (y', xy' - y, x).$$

The subgroup generated by  $\Gamma_1$  consists of four discrete point symmetries, which can be found without having to consider contact symmetries [7]. The four inequivalent non-point contact symmetries are obtained from the point symmetries by composition with  $\Gamma_2$ , which is the prolonged Legendre Transform.

Generally speaking, the larger the dimension of  $\mathcal{L}$ , the easier it is to find the discrete symmetries. Nevertheless, the contact condition makes it possible to solve the governing equations, even when  $N = 1$ . Suppose that  $\mathcal{L}$  is one-dimensional and the ODE is written in canonical coordinates, so that the continuous symmetries are generated by

$$X = \partial_y.$$

Then Lemma 1 gives  $X = b\hat{X}$ , where  $b \neq 0$ , and therefore

$$\hat{x} = f(x, y'), \quad \hat{y} = by + g(x, y').$$

The contact condition yields

$$\hat{y}' = h(x, y'),$$

where

$$g_x = f_x h - by', \quad g_{y'} = f_{y'} h. \quad (3.6)$$

Equations (3.6) are compatible if and only if

$$f_x h_{y'} - f_{y'} h_x = b. \quad (3.7)$$

They can be integrated, once  $f$  and  $h$  are known, to determine  $g$  up to an arbitrary constant, which may be set at any convenient value to factor out equivalence under the one-parameter group generated by  $X$ .

Consider the general third order ODE admitting the group generated by  $X$ :

$$y''' = \omega(x, y', y'').$$

Substituting

$$\hat{x} = f(x, y'), \quad \hat{y}' = h(x, y')$$

into the symmetry condition, and then equating powers of  $y''$ , yields an over-determined coupled system of nonlinear PDEs. This system is precisely as intractable as the problem of using the symmetry condition alone to find all point symmetries of

$$y'' = \omega(x, y, y').$$

However, with the aid of the contact condition (3.7), the problem simplifies considerably.

To illustrate this, consider the ODE

$$y''' = y''^3 \sin\left(\frac{x}{y'}\right), \quad (3.8)$$

whose only Lie contact symmetries are those generated by  $X = \partial_y$ . The symmetry condition gives the over-determined (but complicated) system

$$f_x h_{xx} - f_{xx} h_x = h_x^3 \sin\left(\frac{f}{h}\right),$$

$$f_{y'} h_{xx} + 2f_x h_{xy'} - f_{xx} h_{y'} - 2f_{xy'} h_x = 3h_x^2 h_{y'} \sin\left(\frac{f}{h}\right),$$

$$f_x h_{y'y'} + 2f_{y'} h_{xy'} - f_{y'y'} h_x - 2f_{xy'} h_{y'} = 3h_x h_{y'}^2 \sin\left(\frac{f}{h}\right),$$

$$f_{y'} h_{y'y'} - f_{y'y'} h_{y'} + (f_x h_{y'} - f_{y'} h_x) \sin\left(\frac{x}{y'}\right) = h_{y'}^3 \sin\left(\frac{f}{h}\right).$$

This system can be greatly simplified by using (3.7) and its differential consequences, which reduces the first three equations to

$$f_{xx} = f_{xy'} = h_x = h_{y'y'} = 0.$$

Combining this result with (3.7) and the remaining symmetry condition gives

$$f = \alpha(x + 2n\pi y'), \quad h = y', \quad \alpha \in \{-1, 1\}, \quad n \in \mathbb{Z}.$$

After solving (3.6) for  $g(x, y')$  and setting  $g(0, 0) = 0$  to factor out the continuous symmetries, we obtain the following result. The inequivalent discrete contact symmetries of (3.8) form a countably-infinite group, which is generated by

$$\Gamma_1 : (x, y, y') \mapsto (-x, -y, y'),$$

$$\Gamma_2 : (x, y, y') \mapsto (x + 2\pi y', y + \pi y'^2, y').$$

## 4. Discrete uniform contact symmetries

For ODEs of order  $n \geq 3$ , the Lie contact symmetries can generally be found systematically, and  $\mathcal{L}$  is finite-dimensional. Second-order ODEs have an infinite-dimensional Lie algebra of contact symmetry generators, but they cannot all be found unless the general solution of the ODE is known. However, some contact symmetries may be found with the aid of a suitable ansatz for  $Q$ . For example, the restriction  $Q_{y'y'} = 0$  enables all Lie point symmetries to be found systematically.

Other restrictions on  $Q$  are possible. For example, the set of all *uniform* contact symmetries of a given ODE of order  $n \geq 2$  is a finite-dimensional Lie group [8]. Uniform contact symmetries are of the form

$$\hat{x} = \Phi(x, y'), \quad \hat{y} = ky + \Theta(x, y'), \quad \hat{y}' = \Psi(x, y'), \quad k \in \mathbb{R} \setminus \{0\}, \quad (4.1)$$

where the contact condition requires that

$$ky' + \Theta_x = \Phi_x \Psi, \quad \Theta_{y'} = \Phi_{y'} \Psi. \quad (4.2)$$

Many of the most commonly-occurring contact symmetries are uniform, including all contact symmetries of the ODEs that were used as examples in the previous section. The generators of uniform Lie contact symmetries have characteristic functions of the form

$$Q = c_1 y + \phi(x, y'), \quad c_1 \in \mathbb{R}.$$

The Lie algebra of these generators can be found systematically, for a given ODE of order  $n \geq 2$ , by equating powers of  $y$  in the symmetry condition. The constructions leading to Lemma 1 can be repeated, restricting attention to uniform contact symmetries, to obtain the following.

**Lemma 2.** *Every uniform contact symmetry  $\Gamma$  of an ordinary differential equation of order  $n \geq 2$  induces an automorphism of the Lie algebra,  $\mathcal{L}$ , of generators of one-parameter local Lie groups of uniform contact symmetries of the differential equation. For each such  $\Gamma$ , there exists a constant non-singular matrix  $(b_i^l)$  such that*

$$X_i = b_i^l \hat{X}_l. \quad (4.3)$$

*This automorphism preserves all structure constants.*

This lemma can be used to find all discrete uniform contact symmetries in exactly the same way as Lemma 1 is used. To illustrate this, consider the ODE

$$y'' = \frac{y'^2}{3xy' - 4y}, \quad (4.4)$$

which has a four-dimensional Lie algebra of uniform contact symmetry generators. The structure constants are simplest in the basis

$$X_1 = 2y'\partial_x + y'^2\partial_y,$$

$$X_2 = \frac{3}{4}x\partial_x + \frac{1}{2}y\partial_y - \frac{1}{4}y'\partial_{y'},$$

$$X_3 = 2(y')^{-3}\partial_x + 3(y')^{-2}\partial_y,$$

$$X_4 = \frac{1}{4}x\partial_x + \frac{1}{2}y\partial_y + \frac{1}{4}y'\partial_{y'}.$$

The only non-zero structure constants are

$$c_{12}^1 = -c_{21}^1 = 1, \quad c_{34}^3 = -c_{43}^3 = 1,$$

and so the Lie algebra is isomorphic to  $\mathfrak{a}(1) \oplus \mathfrak{a}(1)$ . After simplifying  $B$  as far as possible, using the relations (2.16) and the adjoint action of the continuous group, we obtain two possibilities. Either

$$B = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \alpha, \beta \in \{-1, 1\}, \quad (4.5)$$

or

$$B = \begin{bmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ \beta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \alpha, \beta \in \{-1, 1\}. \quad (4.6)$$

The inequivalent discrete uniform contact symmetries are calculated in the same way as previously. Lemma 2 is used, together with (4.1) and the contact condition (4.2), to obtain the most general form possible for a uniform contact symmetry of (4.4). The symmetry condition for the ODE is then used to determine which of the possible solutions actually are symmetries. We find that the group of inequivalent discrete uniform symmetries of (4.4) is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ , and is generated by

$$\Gamma_1 : (x, y, y') \mapsto (-x, y, -y'),$$

$$\Gamma_2 : (x, y, y') \mapsto (x, -y, -y'),$$

$$\Gamma_3 : (x, y, y') \mapsto (xy'^2, 2xy' - y, (y')^{-1}).$$

The discrete point symmetries generated by  $\Gamma_1$  and  $\Gamma_2$  can also be found directly from the Lie algebra of point symmetries, which is  $\text{Span}(X_2, X_4)$ . These inequivalent discrete symmetries are derived from the matrix  $B$  in (4.5), with  $\alpha = \beta$ ; they map each of the  $\mathfrak{a}(1)$  Lie subalgebras to itself. However  $\Gamma_3$ , which is derived from (4.6), interchanges these two subalgebras. Actually, (4.4) is merely one representative of a whole class of second-order ODEs that have inequivalent uniform discrete symmetries  $\Gamma_i$ ,  $i = 1, 2, 3$ ; these ODEs are of the form

$$y'' = \frac{my'^2}{y - (m+1)xy'}, \quad m \in \mathbb{R} \setminus \{0\}.$$

## 5. Contact symmetries of a nonlinear PDE

The method described in section 2 generalizes to PDEs without difficulty; the corresponding algorithm for point symmetries will be discussed elsewhere [9]. The basic steps are shown here, using the potential hyperbolic heat equation

$$u_{tt} + u_t = \frac{u_{xx}}{u_x} \quad (5.1)$$

as an example. The Lie algebra of contact symmetry generators is five-dimensional, with a basis

$$\begin{aligned} X_1 &= \partial_t, & X_2 &= \partial_x, & X_3 &= \partial_u, & X_4 &= e^{-t}\partial_u - e^{-t}\partial_{u_t}, \\ X_5 &= -x\partial_x + u\partial_u + u_t\partial_{u_t} + 2u_x\partial_{u_x}. \end{aligned}$$

These Lie contact symmetries are actually point symmetries, but there is still the possibility that some discrete contact symmetries may be non-point symmetries, *cf.* (3.1). Using (2.16) and the adjoint action of the continuous group to simplify  $B$ , we obtain two possibilities. Either

$$B = \text{diag}\{1, b, \alpha, \beta, 1\}, \quad \alpha, \beta \in \{-1, 1\}, \quad b \in \mathbb{R} \setminus \{0\}, \quad (5.2)$$

or

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 2 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha, \beta \in \{-1, 1\}, \quad b \in \mathbb{R} \setminus \{0\}. \quad (5.3)$$

If  $B$  is of the form (5.2), the equations analogous to (2.13), (2.14) are

$$\begin{bmatrix} X_1\hat{t} & X_1\hat{x} & X_1\hat{u} \\ X_2\hat{t} & X_2\hat{x} & X_2\hat{u} \\ X_3\hat{t} & X_3\hat{x} & X_3\hat{u} \\ X_4\hat{t} & X_4\hat{x} & X_4\hat{u} \\ X_5\hat{t} & X_5\hat{x} & X_5\hat{u} \end{bmatrix} = B \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & e^{-\hat{t}} \\ 0 & -\hat{x} & \hat{u} \end{bmatrix},$$

whose general solution is

$$\hat{t} = t + c_1, \quad \hat{x} = bx + c_2|u_x|^{-\frac{1}{2}}, \quad \hat{u} = \alpha u + (\alpha - \beta e^{-c_1})u_t + c_3|u_x|^{\frac{1}{2}}, \quad c_i \in \mathbb{R}.$$

The contact condition and the symmetry condition reduce these further, to

$$(\hat{t}, \hat{x}, \hat{u}) = (t, \alpha x, \alpha u), \quad \alpha \in \{-1, 1\}.$$

The remaining inequivalent discrete contact symmetries are obtained from (5.3) in a similar way. To summarize the results: the inequivalent discrete contact symmetries of (5.1) form a group isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , which is generated by

$$\Gamma_1 : (t, x, u, u_t, u_x) \mapsto (t, -x, -u, -u_t, u_x),$$

$$\Gamma_2 : (t, x, u, u_t, u_x) \mapsto (t + \ln|u_x|, u + u_t, x + u_t(u_x)^{-1}, -u_t(u_x)^{-1}, (u_x)^{-1}).$$

The symmetry  $\Gamma_2$  was known previously [15], but it was not known that  $\Gamma_2$  and  $\Gamma_1\Gamma_2$  are the only non-point contact symmetries (up to equivalence). The method outlined in the current paper enables the user to completely classify all contact symmetries of a given differential equation with a known non-trivial Lie algebra.

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