On the Analytical Approach to the $N$-Fold Bäcklund Transformation of Davey-Stewartson Equation

S.K. PAUL and A. ROY CHOWDHURY

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700032, India

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Abstract

$N$-fold Bäcklund transformation for the Davey-Stewartson equation is constructed by using the analytic structure of the Lax eigenfunction in the complex eigenvalue plane. Explicit formulae can be obtained for a specified value of $N$. Lastly it is shown how generalized soliton solutions are generated from the trivial ones.

Introduction: Inverse scattering transform holds a central place in the analysis of nonlinear integrable system in either (1+1)- or (2+1)-dimensions [1]. On the other hand it has been found that explicit soliton solutions can also be obtained by the use of Bäcklund transformations in a much easier way [2]. These Bäcklund transformations are also useful in proving the superposition formulae for these solutions. There have been many attempts to construct explicit $N$-soliton solutions for nonlinear integrable system either by Bäcklund transformations or the inverse scattering method. A separate and elegant approach was developed by Zakharov et al. [3] which relied on the pole structure of the Lax eigenfunction and use of projection operators. In this letter we have used an approach similar to that of Zakharov et al but have generated a formulae for the $N$-fold BT of the nonlinear field variables occurring in the (2+1)-dimensional Davey-Stewartson equation [4]. Our approach is very similar to that of gauge transformation repeatedly applied to any particular seed solution. Lastly we demonstrate how non-trivial solutions are generated by starting with known trivial ones.

Formulation: The Davey-Stewartson equation under consideration can be written as

$$
\begin{align*}
ir_t + r_{xx} - r_{yy} + r(A_2 - A_1) &= 0, \\
iq_t + q_{yy} - q_{xx} + q(A_1 - A_2) &= 0, \\
A_{1x} &= -\frac{1}{2}(q yr + r yq), \\
A_{2y} &= -\frac{1}{2}(q xr + r xq).
\end{align*}
$$

(1)
Equation (1) is known to be a result of the consistency of the operators $T_1$ and $T_2$ written as $[T_1, T_2] \Psi = 0$, where
\[
T_1 \Psi = \left\{ 2 \left( \begin{array}{cc} \partial_x & 0 \\ 0 & \partial_y \end{array} \right) + \left( \begin{array}{c} 0 \\ r \end{array} \right) \right\} \Psi = 0, \\
T_2 \Psi = \frac{1}{2} \left\{ (i \partial_t + \partial_x^2 + \partial_y^2) + \left( \begin{array}{cc} 0 & q_x \\ r_y & 0 \end{array} \right) + A \right\} \Psi = -\frac{K^2}{2} \Psi,
\]
and $A = \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right)$. $K = \text{const}$.

To proceed further we set $A_1 = f_{1y}$, $A_2 = f_{2x}$

and
\[
\Psi = \Phi \exp \left\{ i \left( \alpha + \lambda^{-2} \right) x - i \left( \beta - \lambda^{-2} \right) y \right\}.
\]

Whence the Lax pair becomes,
\[
M \Phi = U \Phi, \quad \Phi_t = V \Phi
\]

with
\[
M = \left( \begin{array}{cc} \partial_x & 0 \\ 0 & \partial_y \end{array} \right), \quad U = \left( \begin{array}{cc} -i \left( \alpha + \lambda^{-2} \right) & -q/2 \\ -r/2 & i \left( \beta - \lambda^{-2} \right) \end{array} \right)
\]

and
\[
V = \left( \begin{array}{cc} i \Lambda + Q(\partial_x, \partial_y) + i f_{1y} \\ i r_y \end{array} \begin{array}{cc} i q_x \\ i \Lambda + Q(\partial_x, \partial_y) + i f_{2y} \end{array} \right).
\]

With
\[
\Lambda = K^2 - \left( \alpha + \lambda^{-2} \right)^2 - \left( \beta - \lambda^{-2} \right)^2, \\
Q(\partial_x, \partial_y) = i \left( \partial_x^2 + \partial_y^2 \right) - 2 \left\{ \left( \alpha + \lambda^{-2} \right) \partial_x - \left( \beta - \lambda^{-2} \right) \partial_y \right\},
\]

we can construct particular Jost solutions, corresponding to $q = q_0 = \text{const}$, $r = r_0 = \text{const}$, $A_1 = A_{10} = f_{1y}^0$ and $A_2 = A_{20} = f_{2x}^0$ with $f_{1y}^0 = f_{2x}^0 = \text{const}$. This particular eigenvector $\Phi_0$ turns out to be
\[
\hat{\Phi}_0 = \left( \begin{array}{cc} \exp(\theta_1 x + \chi_1 y + \xi_1 t) \\ m_0 \exp(\theta_1 x + \chi_1 y + \xi_1 t) \end{array} \begin{array}{cc} \exp(\theta_2 x + \chi_2 y + \xi_2 t) \\ n_0 \exp(\theta_2 x + \chi_2 y + \xi_2 t) \end{array} \right)
\]

with
\[
\theta_1 = -i \left( \alpha + \lambda^{-2} \right) + a m_0, \quad \theta_2 = -i \left( \alpha + \lambda^{-2} \right) + a n_0, \\
\chi_1 = b / m_0 + i \left( \beta - \lambda^{-2} \right), \quad \chi_2 = b / n_0 + i \left( \beta - \lambda^{-2} \right),
\]

$\xi_1, \xi_2$ are arbitrary complex constants, $m_0, n_0$ are arbitrary constants and $m_0 \neq n_0$.

Note that $\det \hat{\Phi}_0 \neq 0$ so that $\Phi_0^{-1}$ exists.
Now suppose that \( \Phi_{n-1} \) denotes the Lax eigenfunction, corresponding to the \((n-1)\) soliton solution, and \( B_n(x,y,t) \) be the transformation which yields the \( \Phi_n \) (solution corresponding to the \( n \) soliton case when applied to \( \Phi_{n-1} \)), that is

\[
\Phi_n(x, y, t, \lambda) = B_n(x, y, t, \lambda) \Phi_{n-1}(x, y, t, \lambda).
\]

Using the above Lax equations (2) and (4), we can at once deduce the equations satisfied by \( B_n \)

\[
MB_n = U_n B_n - B_n U_{n-1}, \quad \partial_t B_n = V_n B_n - B_n V_{n-1},
\]

where \( U_n, V_n \) denote the Lax matrices.

Corresponding to the \( n \)-soliton solution: Note that \( U, V \) are even functions of \( \lambda \), so that we can assume that

\[
B_n(-\lambda) = B_n(\lambda).
\]

We now assume \( B_n \) to have simple pole structure in the complex \( \lambda \)-plane, so that

\[
B_n(x, y, t, \lambda) = Q_n + \frac{2\lambda}{\lambda^2 - \lambda_n^2} P_n.
\]

We also assume that

\[
B_n^{-1}(x, y, t, \lambda) = Q'_n + \frac{2\lambda'}{\lambda^2 - \lambda_n^2} P'_n,
\]

where \( P_n, Q_n, P'_n, Q'_n \) are matrix functions of \((x, y, t)\). The condition \( B_n B_n^{-1} = B_n^{-1} B_n = I \) leads to

\[
B_n(\lambda') P'_n = 0, \quad P_n B_n^{-1}(\lambda_n) = 0,
\]

\[
B_n^{-1}(\lambda_n) P_n = 0, \quad P'_n B_n(\lambda'_n) = 0.
\]

**Calculation of the matrices** \( Q_n, P_n, Q'_n, P'_n \): Let us now go back to equation (5) and use the expression (6) and (7). Rewriting \( U_n \) as

\[
U_n = -i\lambda^{-2}I + U'_n
\]

and using the form of \( B_n \) given in (6) we get:

\[
MQ_n + \frac{2\lambda_n}{\lambda^2 - \lambda_n^2} (MP_n) = (-i\lambda^{-2}I + U'_n) \left( Q_n + \frac{2\lambda}{\lambda^2 - \lambda_n^2} P_n \right)
\]

\[
- \left( Q_n + \frac{2\lambda}{\lambda^2 - \lambda_n^2} P_n \right) (-i\lambda^{-2}I + U'_{n-1})
\]

which yields equations satisfied by \( P_n \) and \( Q_n \):

\[
M(P_n \Phi_{n-1}(\lambda_n)) = U_n(\lambda_n)(P_n \Phi_{n-1}(\lambda_n)),
\]

\[
M(Q_n \Phi_{n-1}(\lambda_n)) = U_n(\lambda_n)(Q_n \Phi_{n-1}(\lambda_n)).
\]

Using the same form of \( V_n \) as give in equation (3) in the time part, we get the following equations:

\[
\partial_x \left( Q_n - \frac{2}{\lambda_n} P_n \right) = -\partial_y \left( Q_n - \frac{2}{\lambda_n} P_n \right),
\]

\[
\partial_t (P_n \Phi_{n-1}(\lambda_n)) = V_n(\lambda_n)(P_n \Phi_{n-1}(\lambda_n)),
\]

\[
Q_{nt} = OP_2 Q_n + 2i(Q_{nx} \partial_x + Q_{ny} \partial_y) + D_n Q_n - Q_n D_{n-1},
\]
where operator $OP_2 \equiv i(\partial_x^2 + \partial_y^2) - 2(\alpha \partial_x - \beta \partial_y)$, $\alpha, \beta = \text{const.}$, and $D_n = \begin{pmatrix} if_1^n & iq_{nx} \\ ir_{ny} & if_2^n \end{pmatrix}$.

As per the ansatz of Zakharov we search for $P_n$ in the form,

$$P_n = \begin{pmatrix} \gamma_{n1} \\ \gamma_{n2} \end{pmatrix} (\delta_{n1}, \delta_{n2}).$$

It is interesting to observe that

$$(\delta_{n1}, \delta_{n2}) = (a_{n1}, a_{n2})\Phi_{n-1}^{-1}(n),$$

where $a_{n1}, a_{n2}$ are practically two constants.

Similarly for $P'_n$ and $Q'_n$, we set

$$P'_n = \begin{pmatrix} \gamma'_{n1} \\ \gamma'_{n2} \end{pmatrix} (\delta'_{n1}, \delta'_n) \quad \text{and} \quad Q'_n = \begin{pmatrix} \alpha'_{n1} & \alpha''_{n1} \\ \beta'_{n1} & \beta''_{n1} \end{pmatrix}.$$ 

Whence we get

$$B_i(\lambda'_i) = \begin{pmatrix} F^{11}_i + \sigma_i \gamma_{i1} \delta_{i1} & F^{12}_i + \sigma_i \gamma_{i2} \delta_{i2} \\ F^{21}_i + \sigma_i \gamma_{i2} \delta_{i1} & F^{22}_i + \sigma_i \gamma_{i2} \delta_{i2} \end{pmatrix},$$

where

$$F^{11}_i = f^{11}_{i1}(t) \exp\{im_{i1}(x-y)\}, \quad F^{12}_i = f^{12}_{i1}(t) \exp\{im'_{i1}(x-y)\},$$

$$F^{21}_i = f^{21}_{i2}(t) \exp\{im_{i2}(x-y)\}, \quad F^{22}_i = f^{22}_{i2}(t) \exp\{im'_{i2}(x-y)\},$$

and

$$\sigma_i = \frac{2}{\lambda_i} - \frac{2\lambda_i}{\lambda_i^2 - \lambda_i'^2},$$

$$\delta_{i1} = -\frac{\delta'_{i1} F^{11}_i + \delta'_{i2} F^{21}_i}{\sigma_i (\delta'_{i1} \gamma_{i1} + \delta'_{i2} \gamma_{i2})}, \quad \delta_{i2} = -\frac{\delta'_{i1} F^{12}_i + \delta'_{i2} F^{22}_i}{\sigma_i (\delta'_{i1} \gamma_{i1} + \delta'_{i2} \gamma_{i2})},$$

$$\gamma_{i1} = -\frac{\gamma'_{i1} F^{11}_i + \gamma'_{i2} F^{21}_i}{\sigma_i (\gamma'_{i1} \delta_{i1} + \gamma'_{i2} \delta_{i2})}, \quad \gamma_{i2} = -\frac{\gamma'_{i1} F^{21}_i + \gamma'_{i2} F^{22}_i}{\sigma_i (\gamma'_{i1} \delta_{i1} + \gamma'_{i2} \delta_{i2})}.$$ 

$$B_i^{-1}(\lambda'_i) = \begin{bmatrix} \alpha'_{i1} + \epsilon'_{i1} \gamma'_{i1} \delta_{i1} & \alpha''_{i1} + \epsilon'_{i1} \gamma'_{i1} \delta'_{i1} \\ \beta'_{i1} + \epsilon'_{i1} \gamma'_{i2} \delta_{i1} & \beta''_{i1} + \epsilon'_{i1} \gamma'_{i2} \delta'_{i1} \end{bmatrix}.$$ 

Here $\epsilon'_{i} = \frac{2\lambda'}{\lambda_i^2 - \lambda_i'^2}$, along with

$$\delta'_{i1} = -\frac{\alpha'_{i1} \delta_{i1} + \beta'_{i2} \delta_{i2}}{\epsilon'_{i} (\delta_{i1} \gamma'_{i1} + \delta_{i2} \gamma_{i2})}, \quad \delta'_{i2} = -\frac{\alpha''_{i1} \delta_{i1} + \beta''_{i2} \delta_{i2}}{\epsilon'_{i} (\delta_{i1} \gamma'_{i1} + \delta_{i2} \gamma_{i2})},$$

$$\gamma'_{i1} = -\frac{\gamma'_{i1} \alpha'_{i1} + \gamma'_{i2} \alpha''_{i1}}{\epsilon'_{i} (\gamma_{i1} \delta_{i1} + \gamma_{i2} \delta_{i2})}, \quad \gamma'_{i2} = -\frac{\gamma'_{i1} \beta'_{i1} + \gamma'_{i2} \beta''_{i1}}{\epsilon'_{i} (\gamma_{i1} \delta_{i1} + \gamma_{i2} \delta_{i2})}.$$ 

Finally the matrix $Q_i$ is given as

$$Q_i = \begin{pmatrix} f^{11}_{i1}(t) \exp\{im_{i1}(x-y)\} + \frac{2}{\lambda_i} \gamma_{i1} \delta_{i1} & f^{12}_{i1}(t) \exp\{im'_{i1}(x-y)\} + \frac{2}{\lambda_i} \gamma_{i1} \delta_{i2} \\ f^{21}_{i2}(t) \exp\{im_{i2}(x-y)\} + \frac{2}{\lambda_i} \gamma_{i2} \delta_{i1} & f^{22}_{i2}(t) \exp\{im'_{i2}(x-y)\} + \frac{2}{\lambda_i} \gamma_{i2} \delta_{i2} \end{pmatrix}.$$
It is also very convenient to rewrite the matrix elements of $Q$ and $P$ in terms of Lax eigenfunctions $\Phi$. We collect these results below without giving the detailed derivation,

\[
\begin{aligned}
Q^{11}_l &= \frac{R_l \Phi^{21}_{l-1}(\lambda_l) + R'_l \Phi^{22}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)}, & Q^{12}_l &= \frac{M_l \Phi^{21}_{l-1}(\lambda_l) + M'_l \Phi^{22}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)}, \\
Q^{21}_l &= \frac{L_l \Phi^{21}_{l-1}(\lambda_l) + L'_l \Phi^{22}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)}, & Q^{22}_l &= \frac{N_l \Phi^{21}_{l-1}(\lambda_l) + N'_l \Phi^{22}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)},
\end{aligned}
\]

where

\[
\begin{aligned}
R_l &= -\frac{\lambda^2_l b_l f^{11}_l(t) \exp\{im_{11}(x-y)\}}{\lambda^2_l}, \\
R'_l &= \frac{1}{\lambda^2_l} \left\{ \lambda^2_l a_l f^{11}_l(t) \exp\{im_{11}(x-y)\} + (\lambda^2_l - \lambda^2_l^2) b_l f^{12}_l(t) \exp\{im_{11}'(x-y)\} \right\}, \\
M_l &= \frac{1}{\lambda^2_l} \left\{ - (\lambda^2_l - \lambda^2_l) a_l f^{11}_l(t) \exp\{im_{11}(x-y)\} - \lambda^2_l b_l f^{12}_l(t) \exp\{im_{11}'(x-y)\} \right\}, \\
M'_l &= \frac{\lambda^2_l a_l f^{22}_l(t) \exp\{im_{11}(x-y)\}}{\lambda^2_l}, & L_l &= \frac{\lambda^2_l b_l f^{21}_l(t) \exp\{im_{22}(x-y)\}}{\lambda^2_l}, \\
L'_l &= \frac{1}{\lambda^2_l} \left\{ \lambda^2_l a_l f^{21}_l(t) \exp\{im_{22}(x-y)\} + (\lambda^2_l - \lambda^2_l) b_l f^{22}_l(t) \exp\{im_{22}'(x-y)\} \right\}, \\
N_l &= \frac{1}{\lambda^2_l} \left\{ - (\lambda^2_l - \lambda^2_l) a_l f^{21}_l(t) \exp\{im_{22}(x-y)\} - \lambda^2_l b_l f^{22}_l(t) \exp\{im_{22}'(x-y)\} \right\}, \\
N'_l &= \frac{\lambda^2_l a_l f^{22}_l(t) \exp\{im_{22}(x-y)\}}{\lambda^2_l}.
\end{aligned}
\]

The elements of the $P_l$ matrix are:

\[
\begin{aligned}
P^{11}_l &= -\frac{F_l}{\sigma_l} \frac{\Phi^{22}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)}, & P^{12}_l &= \frac{F_l}{\sigma_l} \frac{\Phi^{21}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)}, \\
P^{21}_l &= -\frac{F'_l}{\sigma_l} \frac{\Phi^{22}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)}, & P^{22}_l &= \frac{F'_l}{\sigma_l} \frac{\Phi^{21}_{l-1}(\lambda_l)}{a_l \Phi^{22}_{l-1}(\lambda_l) - b_l \Phi^{21}_{l-1}(\lambda_l)},
\end{aligned}
\]

with

\[
\begin{aligned}
F_l &= a_l f^{11}_l(t) \exp\{im_{11}(x-y)\} + b_l f^{12}_l(t) \exp\{im_{11}'(x-y)\}, \\
F'_l &= a_l f^{11}_l(t) \exp\{im_{11}(x-y)\} + b_l f^{12}_l(t) \exp\{im_{11}'(x-y)\}, \\
\sigma_l &= \frac{2\lambda^2_l}{\lambda_l \left( \lambda^2_l - \lambda^2_l^2 \right)}.
\end{aligned}
\]

**Construction of the nonlinear fields:** Once the form of the matrices $P_l$ and $Q_l$ are determined we can construct the matrix $B_l$, so that the Lax eigenfunction $\Phi_l(\lambda)$ for the next stage can be determined from that of the previous one via,

\[
\Phi_l(\lambda) = B_l(\lambda) \Phi_{l-1}(\lambda).
\]
These expressions are very complicated, so we just quote one of them to display their structure. For example,

\[
\begin{align*}
\Phi_{1}^{11}(\lambda) &= N_{1}/D_{1}, \\
D_{1} &= a_{t}\Phi_{l-1}^{21}(\lambda_{t}) - b_{t}\Phi_{l-1}^{22}(\lambda_{t}), \\
N_{t} &= R_{t}\Phi_{l-1}^{21}(\lambda_{t})\Phi_{l-1}^{11}(\lambda) + \{R_{t} - f_{t}(\lambda)F_{t}\}\Phi_{l-1}^{22}(\lambda_{t})\Phi_{l-1}^{11}(\lambda) \\
&\quad + \{M_{t} + f_{t}(\lambda)F_{t}\}\Phi_{l-1}^{21}(\lambda_{t})\Phi_{l-1}^{22}(\lambda_{t}) + M_{t}\Phi_{l-1}^{22}(\lambda_{t})\Phi_{l-1}^{22}(\lambda_{t}),
\end{align*}
\]  

(8)

with similar expression for other elements \(\Phi_{1}^{12}, \Phi_{2}^{21}\) and \(\Phi_{2}^{22}\).

Now, for the determination of nonlinear fields, consider

\[
U_{n}^{'}, Q_{n} = M_{n}Q_{n} + Q_{n}U_{n-1}^{'},
\]

where

\[
M = \begin{pmatrix} \partial_{x} & 0 \\ 0 & \partial_{y} \end{pmatrix}, \quad U_{n}^{'} = \begin{pmatrix} -i\alpha & -\frac{q_{n}}{2} \\ -\frac{r_{n}}{2} & i\beta \end{pmatrix}, \quad U_{n-1}^{'} = \begin{pmatrix} -i\alpha & -\frac{q_{n-1}}{2} \\ -\frac{r_{n-1}}{2} & i\beta \end{pmatrix}.
\]

In the expression for \(U_{n}^{'}\) and \(U_{n-1}^{'}\) we take \(\alpha = \beta = 0\), which at once yields

\[
q_{n} = -\frac{2Q_{n_{x}}^{12}}{Q_{n}^{11}} + \frac{Q_{n}^{11}}{Q_{n}^{11}}q_{n-1}.
\]

(9)

This is nothing but a simple recursion relation. Similarly,

\[
r_{n} = -\frac{2Q_{n}^{21}}{Q_{n}^{11}} + \frac{Q_{n}^{22}}{Q_{n}^{11}}r_{n-1}.
\]

(10)

Explicitly are can write,

\[
\begin{align*}
n = 1, \quad q_{1} &= -\frac{2Q_{1_{x}}^{12}}{Q_{1}^{11}} + \frac{Q_{1}^{11}}{Q_{1}^{11}}q_{0}, \quad r_{1} = -\frac{2Q_{1_{y}}^{21}}{Q_{1}^{11}} + \frac{Q_{1}^{22}}{Q_{1}^{11}}r_{0}, \\
n = 2, \quad q_{2} &= -\frac{2Q_{2_{x}}^{12}}{Q_{2}^{22}} - \frac{2Q_{2}^{11}Q_{2_{x}}^{12}}{Q_{2}^{11}Q_{2}^{22}} + \frac{Q_{2}^{11}Q_{2}^{11}}{Q_{2}^{22}Q_{2}^{22}}q_{0}, \\
r_{2} &= -\frac{2Q_{2_{y}}^{21}}{Q_{2}^{11}} - \frac{2Q_{2}^{22}Q_{2_{y}}^{11}}{Q_{2}^{22}Q_{2}^{11}} + \frac{Q_{2}^{22}Q_{2}^{22}}{Q_{2}^{22}Q_{2}^{11}}r_{0}.
\end{align*}
\]

So far we have considered \(f_{1}^{11}(t), f_{1}^{12}(t), f_{2}^{21}(t), f_{2}^{22}(t)\) to be functions of time or constants; \(m_{11}, m_{1_{1}}, m_{21}, m_{2_{1}}\) to be arbitrary constants; \(a_{t}, b_{t}\) to be arbitrary constants for all \(l\). Now assume that \(f_{1}^{12}(t) = f_{2}^{21}(t) = 0\) and \(b_{t} = 0\) for all \(l\) values. So that \(R_{t} = M_{t}^{'}, L_{t} = L_{t}^{'}, N_{t} = F_{t}^{'}, F_{t} = 0\) for all \(l\). In this case the form of \(B_{l}(\lambda)\) turns out to be:

\[
B_{l}(\lambda) = \begin{pmatrix} \frac{\lambda_{l}^{2}}{\lambda_{l}^{2}} & \left(1 - \frac{\lambda_{l}^{2}}{\lambda_{l}^{2}}\right) f_{1}^{11}(t)e^{im_{11}(x-y)} \left(1 + \frac{\lambda_{l}^{2}}{\lambda_{l}^{2}}\right) f_{1}^{11}(t)e^{im_{11}(x-y)} \theta_{l-1} \\
0 & f_{2}^{22}(t)e^{im_{22}(x-y)} \end{pmatrix},
\]

where \(\theta_{l-1} = \frac{\Phi_{l-1}^{21}(\lambda_{l})}{\Phi_{l-1}^{22}(\lambda_{l})} \).
To write the formulae for the $n$-soliton solution in a compact form, we note that

$$Q_{n}^{11}Q_{n-1}^{11}Q_{n-2}^{11} \cdots Q_{R}^{11} = A_{n-R+1}^{n}F_{n-R+1}^{n}(t) \exp \left\{ \sum_{j=1}^{n-R+1} i \sum_{j=1}^{n-R+1} m_{1,n-j+1}(x - y) \right\},$$

where $A_{n-R+1}^{n}, F_{n-R+1}^{n}(t)$ stands for

$$A_{n-R+1}^{n} = \prod_{l=1}^{n-R+1} \frac{\lambda_{n-l+1}^{2}}{\lambda_{n-l+1}^{2}}, \quad F_{n-R+1}^{n}(t) = \prod_{l=1}^{n-R+1} f_{n-l+1}^{11}(t).$$

Similar expressions can be written for $Q_{n}^{ij}$ and its products. Using these, we at once obtain:

$$q_{n} = \frac{-2\overline{m}_{n}Q_{n}^{12}}{G_{1}^{n}(t) \exp \{ im'_{2,n}(x - y) \}} + \frac{A_{n}^{n}F_{n}^{n}(t) \exp \left\{ \sum_{j=1}^{n} m_{1,n-j+1}(x - y) \right\}}{G_{1}^{n}(t) \exp \left\{ \sum_{j=1}^{n} m_{2,n-j+1}(x - y) \right\}}\frac{-2 \sum_{K=1}^{n-1} A_{K}^{n}F_{K}^{n}(t) \exp \left\{ \sum_{j=1}^{K} m_{1(n-j+1)}(x - y) \right\}}{G_{K+1}^{n}(t) \exp \left\{ \sum_{j=1}^{K+1} m_{2(n-j+1)}(x - y) \right\}}$$

where we have set

$$\overline{m}_{0} = a(m_{0} - n_{0}), \quad G_{K}^{n}(t) = \prod_{l=1}^{K} f_{n-l+1}^{22}(t), \quad \overline{m}_{n} = -im_{1n} + \overline{m}_{0},$$

$$\overline{m}'_{0} = b(1/m_{0} - 1/n_{0}), \quad F_{K}^{n} = \prod_{l=1}^{K} f_{n-l+1}^{11}(t), \quad T_{n} = \frac{m_{0}}{n_{0}} \left( 1 - \frac{\lambda_{n}^{2}}{\lambda_{n}^{2}} \right),$$

$$Q_{n}^{12} = T_{n}f_{n}^{11}(t) \exp (im_{1n}x) \exp (\overline{m}_{0}x + \overline{m}'_{0}y + \delta t).$$

Here $\delta$ is a complex constant.

**Discussions:** In the above analysis we have demonstrated how the pole type ansatz of Zakharov et al [3] can be used to generate a compact formula for the $N$-fold Bäcklund transformation in the case of the Davey-Stewartson equation. The study yields two main results exhibited in equations (8) and (9). While the equation gives a recursive procedure for the determination of the Lax eigenfunction ($\Phi_{l}$ corresponds to the $l$-soliton state) equation (9) and (10) gives the corresponding recursion relation for the nonlinear fields. We have actually checked that for $n = 1$ one obtains the one soliton solution well known in the literature.
References


