Neumann and Bargmann Systems Associated with an Extension of the Coupled KdV Hierarchy

Zhimin JIANG

Department of Mathematics, Shangqiu Teachers College, Shangqiu 476000, China

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Abstract

An eigenvalue problem with a reference function and the corresponding hierarchy of nonlinear evolution equations are proposed. The bi-Hamiltonian structure of the hierarchy is established by using the trace identity. The isospectral problem is nonlinearized as to be finite-dimensional completely integrable systems in Liouville sense under Neumann and Bargmann constraints.

1 Introduction

A major difficulty in theory of integrable systems is that there is to date no completely systematic method for choosing properly an isospectral problem $\psi_x = M\psi$ so that the zero-curvature representation $M_t - N_x + [M, N] = 0$ is nontrivial. By inserting a reference function into AKNS and WKI isospectral problems, we have obtained successfully two new hierarchies [1, 2].

The coupled KdV hierarchy associated with the isospectral problem

$$\psi_x = M\psi, \quad M = \begin{pmatrix} -\frac{1}{2} \lambda + \frac{1}{2} u & -v \\ 1 & \frac{1}{2} \lambda - \frac{1}{2} u \end{pmatrix}$$

(1.1)

is discussed by D. Levi, A. Sym and S. Wojciechowski [3]. The isospectral problem (1.1) has been nonlinearized as finite-dimensional completely integrable systems in Liouville sense [4].

In this paper, we introduce the eigenvalue problem

$$\psi_x = M\psi, \quad M = \begin{pmatrix} -\frac{1}{2} \lambda + \frac{1}{2} u & -v \\ f(v) & \frac{1}{2} \lambda - \frac{1}{2} u \end{pmatrix},$$

(1.2)

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where \( u \) and \( v \) are two scalar potentials, \( \lambda \) is a constant spectral parameter and \( f(v) \) called reference function is an arbitrary smooth function. The bi-Hamiltonian structure of the corresponding hierarchy is established by using the trace identity [5, 6]. Since the reference function \( f(v) \) in (1.2) can be chosen arbitrarily, many new hierarchies and their Hamiltonian forms are obtained. When \( f = (-v)^\beta \) \((\beta \geq 0)\), the isospectral problem (1.2) is nonlinearized as finite-dimensional completely integrable systems in Liouville sense under Neumann and Bargmann constraints between the potentials and eigenfunctions.

## 2 Preliminaries

Consider the adjoint representation of (1.2)

\[
N_x = MN - NM, \quad N = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-j}
\]  

(2.1)

which leads to

\[
c_0 = b_0 = 0, \quad a_0 = -\frac{1}{2} \alpha \quad \text{(constant)},
\]

(2.2)

\[
c_1 = \alpha f(v), \quad b_1 = -\alpha v, \quad a_1 = 0,
\]

(2.3)

\[
c_2 = \alpha (f'(v)v_x + uf(v)), \quad b_2 = \alpha (v_x - uv), \quad a_2 = -\alpha v f(v),
\]

(2.4)

\[
a_j = -\vartheta^{-1} (vc_j + f(v)b_j),
\]

(2.5)

\[
\begin{pmatrix} c_i \\ b_i \end{pmatrix} = \alpha \left( \begin{pmatrix} f(v) \\ -v \end{pmatrix} \right), \quad \begin{pmatrix} c_{j+1} \\ b_{j+1} \end{pmatrix} = L \begin{pmatrix} c_j \\ b_j \end{pmatrix}, \quad j = 1, 2, \ldots,
\]

(2.6)

where \( \vartheta = \frac{d}{dx}, \vartheta \vartheta^{-1} = \vartheta^{-1} \vartheta = 1 \),

\[
L = \begin{pmatrix} \vartheta + u + 2f \vartheta^{-1}v & 2f \vartheta^{-1}f \\ -2v \vartheta^{-1}v & -\vartheta + u - 2v \vartheta^{-1}f \end{pmatrix}.
\]

It is easy from (1.2) and (2.1) to calculate that

\[
\text{tr} \left( N \frac{\partial M}{\partial \lambda} \right) = -a, \quad \text{tr} \left( N \frac{\partial M}{\partial u} \right) = a, \quad \text{tr} \left( N \frac{\partial M}{\partial v} \right) = -c + f'(v)b.
\]

Noticing the trace identity [5, 6]

\[
\left( \frac{\delta}{\delta u}, \frac{\delta}{\delta v} \right) (-a) = \frac{\partial}{\partial \lambda} (a, -c + f'(v)b),
\]

hence we deduce that

\[
\left( \frac{\delta}{\delta u}, \frac{\delta}{\delta v} \right) H_j = \left( G_{j-2}^{(1)}, G_{j-2}^{(2)} \right), \quad H = \frac{a_j}{j},
\]

(2.7)

where

\[
G_{j-2}^{(1)} = a_j, \quad G_{j-2}^{(2)} = -c_j + f'(v)b_j.
\]

(2.8)
3 The hierarchy and its Hamiltonian structure

Let \( \psi \) satisfy the isospectral problem (1.2) and the auxiliary problem

\[
\psi_t = \mathcal{N}\psi, \quad \mathcal{N} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},
\]

where

\[
A = A_m + \sum_{j=0}^{m-1} a_j \lambda^{m-j}, \quad B = \sum_{j=1}^{m} b_j \lambda^{m-j}, \quad C = \sum_{j=1}^{m} c_j \lambda^{m-j}.
\]

The compatible condition \( \psi_{xt} = \psi_{tx} \) between (1.1) and (3.1) gives the zero-curvature representation

\[
M_t - N_x + [M, N] = 0,
\]

from which we have

\[
\begin{align*}
A_m &= w(\partial + u)c_m + wf'(v)(\partial - u)b_m, \\
\begin{pmatrix} u_t \\ v_t \end{pmatrix} &= \theta_0 L \begin{pmatrix} c_m \\ b_m \end{pmatrix} = \theta_0 \begin{pmatrix} c_{m+1} \\ b_{m+1} \end{pmatrix},
\end{align*}
\]

where

\[
w = \frac{1}{2}(vf'(v) + f)^{-1},
\]

\[
\theta_0 = \begin{pmatrix} 2\partial w & -2\partial wf'(v) \\ 2wf & -2w \end{pmatrix}.
\]

By (2.6) we know that Eqs. (3.2) are equivalent to the hierarchy of nonlinear evolution equations

\[
\begin{align*}
\begin{pmatrix} u_t \\ v_t \end{pmatrix} &= \theta_0 L^m \begin{pmatrix} \alpha f(v) \\ -\alpha v \end{pmatrix}, \quad m = 1, 2, \ldots.
\end{align*}
\]

Let the potentials \( u \) and \( v \) in (1.2) belong to the Schwartz space \( S(-\infty, +\infty) \) over \( (-\infty, +\infty) \). Noticing (2.5) and (2.8) we get

\[
\begin{align*}
\begin{pmatrix} c_j \\ b_j \end{pmatrix} &= \theta_1 \begin{pmatrix} G_{j-2}^{(1)} \\ G_{j-2}^{(2)} \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} -2wf'(v)\partial & -2wf \\ -2w\partial & 2wv \end{pmatrix}.
\end{align*}
\]

Then the recursion relations (2.5), (2.6) and the hierarchy (3.2) can be written as

\[
\begin{align*}
G_{-2} &= -\frac{1}{2}\alpha(1,0)^T, \quad G_{-1} = -\alpha(0,vf'(v) + f)^T, \quad G_0 = -\alpha(vf, uf + uvf'(v))^T, \\
KG_{j-1} &= JG_j, \\
(u_t, v_t)^T &= JG_{m-1} = KG_{m-2},
\end{align*}
\]

where \( J = \theta_0 \theta_1 \) and \( K = \theta_0 L \theta_1 \) are two skew-symmetric operators,

\[
J = \begin{pmatrix} 0 & -2\partial w \\ -2w\partial & 0 \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},
\]
in which
\[
\begin{align*}
K_{11} &= -2\partial - 4\partial w(f'(v) + f'(v)\partial w, \\
K_{12} &= -2\partial w + 4\partial w(f'(v)\partial v - \partial f)w, \\
K_{21} &= -2w\partial - 4w(f\partial - v\partial f'(v))w\partial, \\
K_{22} &= -4w(v\partial f + f\partial v)w. \\
\end{align*}
\]

From (2.7) we obtain the desired bi-Hamiltonian form of (3.7)
\[
\begin{pmatrix}
    u_t \\
    v_t \\
\end{pmatrix} = J \begin{pmatrix}
    \delta \\
    \delta \\
\end{pmatrix} \frac{\delta}{\delta u} H_{m+1} = K \begin{pmatrix}
    \delta \\
    \delta \\
\end{pmatrix} \frac{\delta}{\delta v} H_m. \tag{3.8}
\]

4 Nonlinearization of the isospectral problem

Let $\lambda_j$ and $\psi(x) = (q_j(x), p_j(x))^T$ be eigenvalue and the associated eigenfunction of (1.2).

Through direct verification we know that the functional gradient $\nabla_{(u,v)} \lambda_j = \begin{pmatrix}
    \delta \lambda_j \\
    \delta \lambda_j \\
\end{pmatrix}$ satisfies
\[
\nabla_{(u,v)} \lambda_j = (q_j p_j, -p_j^2 - f'(v)q_j^2), \tag{4.1}
\]
\[
\theta_1 \nabla \lambda_j = \begin{pmatrix}
    p_j^2 \\
    -q_j^2 \\
\end{pmatrix}, \quad L \begin{pmatrix}
    p_j^2 \\
    -q_j^2 \\
\end{pmatrix} = \lambda_j \begin{pmatrix}
    p_j^2 \\
    -q_j^2 \\
\end{pmatrix}, \tag{4.2}
\]
in view of (1.2). Substituting the first expression of (4.2) into the second expression and acting with $\theta_0$ upon once, we have
\[
K \nabla \lambda_j = \lambda_j J \nabla \lambda_j. \tag{4.3}
\]

So, the Lenard operator pair $K, J$ and their gradient series $G_j$ satisfy the basic conditions (3.6) and (4.3) given in Refs. [7, 8] for the nonlinearization of the eigenvalue problem (1.2).

**Proposition 4.1.** When $f(v) = (-v)^\beta$ ($\beta \geq 0$), the isospectral problem (1.2) can be nonlinearized as to be a Neumann system.

In fact, the Neumann constraint $G_{-1}|_{a=1} = \sum_{j=1}^{N} \nabla \lambda_j$ gives
\[
\langle q, p \rangle = 0, \langle p, p \rangle = (\beta + 1)(-v)^\beta + \beta(-1)^{\beta-1}\langle q, q \rangle. \tag{4.4}
\]

By differentiating (4.4) with respect to $x$ and using (1.2), we have
\[
\begin{align*}
u &= \frac{1}{\beta + 1} \left( \frac{\langle Ap, p \rangle}{\langle p, p \rangle} + \beta \frac{\langle Aq, q \rangle}{\langle q, q \rangle} \right), \\
v &= \langle q, q \rangle. \tag{4.5}
\end{align*}
\]
Substituting (4.5) into the equations for the eigenfunctions
\[
\begin{pmatrix}
q_jx \\
p_jx
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2}\lambda_j + \frac{1}{2}u & -v \\
-v\beta & \frac{1}{2}\lambda_j - \frac{1}{2}u
\end{pmatrix} \begin{pmatrix}
q_j \\
p_j
\end{pmatrix}, \quad j = 1, \ldots, N,
\]  
(4.6)
we obtain the Neumann system
\[
\begin{aligned}
q_x &= -\frac{1}{2}\Lambda q - \langle q, q \rangle p + \frac{1}{2(\beta + 1)} \left( \frac{\langle \Lambda p, p \rangle}{\langle p, p \rangle} + \beta \frac{\langle \Lambda q, q \rangle}{\langle q, q \rangle} \right) q, \\
p_x &= \frac{1}{2}\Lambda p + \langle p, p \rangle q - \frac{1}{2(\beta + 1)} \left( \frac{\langle \Lambda p, p \rangle}{\langle p, p \rangle} + \beta \frac{\langle \Lambda q, q \rangle}{\langle q, q \rangle} \right) p,
\end{aligned}
\]  
(4.7)
where \( p = (p_1, \ldots, p_N)^T, \) \( q = (q_1, \ldots, q_N)^T, \) \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N), \) and \( \langle \cdot, \cdot \rangle \) stands for the canonical inner product in \( \mathbb{R}^N. \)

**Proposition 4.2.** When \( f(v) = (-v)^\beta \) (\( \beta \geq 0 \)), the isospectral problem (1.2) can be nonlinearized as to be a Bargmann system.

\[
\begin{aligned}
\langle p, p \rangle &= (-1)^\beta \langle q, q \rangle^\beta, \\
\langle p, q \rangle &= 0.
\end{aligned}
\]  
(4.8)
Substituting (4.8) into (4.6), we obtain the finite-dimensional Hamiltonian system
\[
\begin{aligned}
q_x &= -\frac{1}{2}\Lambda q + \langle q, p \rangle \frac{1}{\beta + 1} p + \frac{1}{2(\beta + 1)} \langle p, p \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} q, \\
p_x &= \frac{1}{2}\Lambda p - \frac{1}{2(\beta + 1)} \langle p, p \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} p + \langle q, p \rangle \frac{1}{\beta + 1} p + \frac{\beta}{2(\beta + 1)} \langle q, q \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} p = -\frac{\partial H}{\partial q}.
\end{aligned}
\]  
(4.9)
The Hamiltonian is
\[
H = -\frac{1}{2} \langle \Lambda q, p \rangle + \frac{1}{2} \langle p, p \rangle \langle q, p \rangle \frac{1}{\beta + 1} - \frac{1}{2} \langle q, q \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} q \beta. \]

5 Integrability of the Neumann system

The Poisson brackets of two functions in symplectic space \( (\mathbb{R}^{2N}, dp \wedge dq) \) are defined as
\[
(F, G) = \sum_{j=1}^{N} \left( \frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right) = \langle F_q, G_p \rangle - \langle F_p, G_q \rangle.
\]
The functions defined by ($m = 0, 1, 2, \ldots$)

\[ F_m = -\frac{1}{2} \langle \Lambda^{m+1} q, p \rangle - \frac{1}{2} \sum_{i+j=m} \langle \Lambda^i q, q \rangle \langle \Lambda^j p, p \rangle \]

are in involution in pairs (see, [9]).

Consider the Moser constraint on the tangent bundle

\[ TS^{N-1} = \left\{ (p, q) \in \mathbb{R}^{2N} | F = \langle q, p \rangle = 0, G = \frac{1}{2(\beta + 1)} (\langle p, p \rangle - (-1)^{\beta} \langle q, q \rangle^{\beta}) = 0 \right\}. \]

Through direct calculations we have

\[ (F, F_m) = 0, \quad (F, G) = \langle p, p \rangle, \]

\[ (F_m, G) = -\frac{1}{2(\beta + 1)} \left( \langle \Lambda^{m+1} p, p \rangle + (-1)^{\beta} \beta \langle q, q \rangle^{\beta-1} \langle \Lambda^{m+1} q, q \rangle \right). \]

Thus the Lagrangian multipliers are

\[ \mu_m = \frac{(F_m, G)}{(F, G)} = -\frac{1}{(\beta + 1)} \left( \frac{\langle \Lambda^{m+1} p, p \rangle}{\langle p, p \rangle} + (-1)^{\beta} \beta \frac{\langle q, q \rangle^{\beta-1}}{\langle p, p \rangle} \langle \Lambda^{m+1} q, q \rangle \right). \]

Since $F = 0$ on the tangent bundle $TS^{N-1}$, the restriction of the canonical equation of $H^* = F_0 - \mu_0 F$ on $TS^{N-1}$ is

\[ \begin{align*}
q_x &= F_{0, p} - \mu_0 F_p|_{TS^{N-1}}, \\
p_x &= -F_{0, q} + \mu_0 F_q|_{TS^{N-1}}
\end{align*} \]

which is exactly the Neumann system (4.7).

**Theorem 5.1.** The Neumann system (4.7) ($TS^{N-1}, dp \wedge dq|_{TS^{N-1}}, H^* = F_0 - \mu_0 F$) is completely integrable in Liouville sense.

**Proof.** Let $F^*_m = F_m - \mu_m F$, $m = 1, \ldots, N - 1$, then it is easy to verify $(F^*_k, F^*_l) = 0$ on $TS^{N-1}$. Hence $\{F^*_m\}$ is an involutive system.

### 6 Integrability of the Bargmann system

Let

\[ \Gamma_k = \sum_{j=1, j \neq k}^N \frac{B_{kj}^2}{\lambda_k - \lambda_j}, \quad (6.1) \]

where $B_{kj} = p_k q_j - p_j q_k$, we have (see Refs. [9, 10])

**Lemma 6.1.**

\[ \langle (q, p), p_i^2 \rangle = 2p_i^2, \quad \langle (q, p), q_i^2 \rangle = -2q_i^2. \quad (6.2) \]
\begin{equation}
\frac{\delta}{\lambda_i - \lambda_k} p_k p_l, \quad \frac{\delta}{\lambda_i - \lambda_k} q_k q_l, \quad \frac{-2B_{lk}}{\lambda_l - \lambda_k} (p_k q_l + q_k p_l).
\end{equation}

Lemma 6.2.

\begin{align}
(\Gamma_k, \Gamma_l) &= (\langle q, p \rangle, \Gamma_l) = (\langle q, p \rangle, q(p_l)) = 0, \\
(p_k^2, p_l^2) &= (\langle q_k, p_l \rangle) = (q_k p_k, q_l(p_l)) = 0, \\
(q_k p_k, p_l^2) &= 2p_k p_l \delta_{kl}, \quad \langle q_k, p_l \rangle = 4q_k p_l \delta_{kl}, \quad \langle q_k, p_l q_l \rangle = 2q_k q_l \delta_{kl}.
\end{align}

Proposition 6.1. Let

\[ E_k = \frac{1}{2} \langle \langle q, p \rangle, p_k^2 \rangle - \frac{1}{2} \langle \langle q, p \rangle, q_k^2 \rangle - \frac{1}{2} \lambda_k q_k p_k - \frac{1}{2} \Gamma_k, \]

the \( E_1, \ldots, E_N \) constitute an \( N \)-involutive system.

**Proof.** Obviously \( (E_k, E_l) = 0 \) for \( k = l \). Suppose \( k \neq l \), in virtue of (6.4)–(6.6) and the property of Poisson bracket in \( (\mathbb{R}^{2N}, dp \wedge dq) \), we have

\begin{align*}
4(E_k, E_l) &= \frac{1}{\beta + 1} p_k^2 \langle q, p \rangle + \frac{1}{\beta + 1} p_l^2 \langle q, p \rangle + \frac{1}{\beta + 1} q_k^2 \langle q, p \rangle, \\
&\quad - \frac{1}{\beta + 1} q_k^2 \langle q, p \rangle, \\
&\quad - \langle q, p \rangle \frac{1}{\beta + 1} \Gamma_k, \\
&\quad + \frac{\beta}{\beta + 1} q_k^2 \langle q, p \rangle, \\
&\quad + \langle q, p \rangle \frac{\beta}{\beta + 1} \Gamma_k.
\end{align*}

Substituting (6.2) and (6.3) into the above equation yields \( (E_k, E_l) = 0 \).

Consider a bilinear function \( Q_z(\xi, \eta) \) on \( \mathbb{R}^N \):

\[ Q_z(\xi, \eta) = \langle (z - \Lambda)^{-1} \xi, \eta \rangle = \sum_{k=1}^{N} \frac{\xi_k \eta_k}{z - \lambda_k} = \sum_{m=0}^{\infty} z^{-m-1} \Lambda^m \langle \Lambda^m \xi, \eta \rangle. \]

The generating function of \( \Gamma_k \) is (see, [9, 10])

\[ \left| \begin{array}{cc}
Q_z(q, q) & Q_z(q, p) \\
Q_z(p, q) & Q_z(p, p)
\end{array} \right| = \sum_{k=1}^{N} \frac{\Gamma_k}{z - \lambda_k}. \]

Hence the generating function of \( E_k \) is

\begin{align}
&\frac{1}{2} \langle \langle q, p \rangle, p_k^2 \rangle Q_z(p, p) - \frac{1}{2} \langle \langle q, p \rangle, q_k^2 \rangle Q_z(q, q) - \frac{1}{2} Q_z(\Lambda q, p) \\
&\quad - \frac{1}{2} \left| \begin{array}{cc}
Q_z(q, q) & Q_z(q, p) \\
Q_z(p, q) & Q_z(p, p)
\end{array} \right| \sum_{k=1}^{N} \frac{E_k}{z - \lambda_k}.
\end{align}

(6.7)
Substituting the Laurent expansion of $Q_z$ and
\[(z - \lambda_k)^{-1} = \sum_{m=0}^{\infty} z^{-m-1} \lambda_k^m\]
in to both sides of (6.7) respectively, we have

**Proposition 6.2.** Let
\[F_m = \sum_{k=1}^{N} \lambda_k^m E_k, \quad m = 0, 1, 2, \ldots\]
then
\[F_0 = \frac{1}{2} \langle q, p \rangle^{1/\delta} \langle p, p \rangle - \frac{1}{2} \langle q, p \rangle^{\frac{\delta}{2}} \langle q, q \rangle - \frac{1}{2} \langle \Lambda q, p \rangle,
\]
\[F_m = \frac{1}{2} \langle q, p \rangle^{1/\delta} \langle \Lambda^m p, p \rangle - \frac{1}{2} \langle q, p \rangle^{\frac{\delta}{2}} \langle \Lambda^m q, q \rangle
- \frac{1}{2} \langle \Lambda^m q, p \rangle
- \frac{1}{2} \sum_{j=1}^{m} \left| \frac{\langle \Lambda^{j-1} q, q \rangle \langle \Lambda^{j-1} q, p \rangle}{\langle \Lambda^{m-j} p, q \rangle \langle \Lambda^{m-j} p, p \rangle} \right|.
\]
Moreover, $(F_k, F_l) = 0$.

Hence we arrive at the following theorem.

**Theorem 6.1.** The Bargmann system defined by (4.9) is completely integrable in Liouville sense in the symplectic manifold $(\mathbb{R}^{2N}, dp \wedge dq)$.

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**References**


