

Manin-Olshansky Triples for Lie Superalgebras

Dimitry LEITES and Alexander SHAPOVALOV

*Department of Mathematics, University of Stockholm, Roslagsv. 101,
Kräftriket hus 6, SE-106 91, Stockholm, Sweden
E-mail: mleites@matematik.su.se*

Received September 18, 1999; Revised November 21, 1999; Accepted February 22, 2000

Abstract

Following V. Drinfeld and G. Olshansky, we construct Manin triples $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$ such that \mathfrak{g} is different from Drinfeld's doubles of \mathfrak{a} for several series of Lie superalgebras \mathfrak{a} which have no even invariant bilinear form (periplectic, Poisson and contact) and for a remarkable exception. Straightforward superization of suitable Etingof–Kazhdan's results guarantee then the uniqueness of q -quantization of our Lie bialgebras. Our examples give solutions to the quantum Yang-Baxter equation in the cases when the classical YB equation has no solutions. To find explicit solutions is a separate (open) problem. It is also an open problem to list (à la Belavin-Drinfeld) all solutions of the classical YB equation for the Poisson superalgebras $\mathfrak{po}(0|2n)$ and the exceptional Lie superalgebra $\mathfrak{k}(1|6)$ which has a Killing-like supersymmetric bilinear form but no Cartan matrix.

1 Introduction

Drinfeld proved [3] that \mathfrak{a} is a Lie bialgebra if and only if it constitutes, together with its dual \mathfrak{a}^* , a *Manin triple* $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$, where $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}^*$ as vector spaces, \mathfrak{g} is a Lie algebra possessing a nondegenerate invariant symmetric bilinear form B such that \mathfrak{a} and \mathfrak{a}^* are isotropic with respect to B . The known examples of forms B arise from the Cartan matrix of \mathfrak{g} in case the Cartan matrix is symmetrizable. Such a form B corresponds to the quadratic Casimir element Δ (which for infinite dimensional \mathfrak{g} belongs, strictly speaking, to a completion of $U(\mathfrak{g})$, rather than to $U(\mathfrak{g})$ itself). Etingof and Kazhdan showed [4] how to q -quantize the bialgebra structure of $U(\mathfrak{a})$ in terms of \mathfrak{g} and Δ and proved that such a quantization is unique.

There are two ways to superize these results and constructions. One is absolutely straightforward. Do not misread us! We only refer to the above-mentioned examples and result from deep and difficult papers [3] and [4]. (Besides, performing the actual implementation of these “straightforward” generalizations is quite a job and, moreover, we encounter several unexpected phenomena, e.g., while presenting simple Lie superalgebras, cf. [6].)

The only totally new feature of “straightforward” superization is the fact that there is just one (in the class of \mathbb{Z} -graded Lie superalgebras of polynomial growth) series of simple Lie superalgebras $\mathfrak{sh}(0|n)$ (see [9]) and one “exception”, $\mathfrak{k}^L(1|6)$, (see [5]) which have no Cartan matrix but, nevertheless, possess an invariant nondegenerate supersymmetric even bilinear form.

Another superization, performed by G. Olshansky, is totally new. Drinfeld observed that the definition of Manin triples does not require a nondegenerate symmetric invariant bilinear form on \mathfrak{a} . For a simple Lie algebra \mathfrak{a} , be it finite dimensional or of polynomial growth, such a form always exists; hence, $\mathfrak{a} \cong \mathfrak{a}^*$ and $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}$, i.e., is the *double* of the bialgebra \mathfrak{a} . G. Olshansky [10] described two series of simple Lie superalgebras \mathfrak{a} without nondegenerate symmetric invariant bilinear form for which there still exist Manin triples. His construction gives solutions to the *quantum* Yang-Baxter equation in the cases the *classical* equation has *no* solutions, compare with the classification [9].

Our result. We list other examples of Manin triples of G. Olshansky type for the known Lie superalgebras of polynomial growth. Their q -quantization is routine thanks to Etingof–Kazhdan’s recipe. (To get explicit formulas for the R -matrix is, though routine, a quite tedious job and we leave it, so far, as an open problem.) Observe that in some of our examples, as well as in [10], \mathfrak{a} is not simple but “close” to a simple one (like affine Kac–Moody algebras).

1.1 On background

For background on linear algebra in superspaces and the list of simple stringy and vectorial Lie superalgebras see [12], [13]. We recall here only the additional data. On a simple finite dimensional Lie algebra \mathfrak{g} there exists only one up to proportionality invariant (with respect to the adjoint action) nondegenerate symmetric bilinear form. In this one-dimensional space of invariant forms one usually chooses for the point of reference the *Killing form* $\langle x, y \rangle_{\text{ad}} = \text{tr}(\text{adx} \cdot \text{ady})$. For any irreducible finite dimensional representation ρ of \mathfrak{g} we have a proportional form $\langle x, y \rangle_{\rho} = \text{tr}(\rho(x) \cdot \rho(y))$.

Remark. In reality, the Killing form is not the easiest to use, but since all of the forms on the simple Lie superalgebra (of polynomial growth or finite dimensional) are proportional to each other, one can take most convenient for the task. For $\mathfrak{sl}(n)$, for example, it is more convenient to take the form associated with the standard representation: $\langle x, y \rangle_{\text{id}} = \text{tr}(x \cdot y)$.

Statement 1.1. ([13]) *An invariant (with respect to the adjoint action) nondegenerate symmetric bilinear form on a simple Lie superalgebra \mathfrak{g} , if exists, is unique up to proportionality.*

Observe that on Lie *superalgebras* the form can be either even or odd, the Killing form can be identically zero even if there is another, nondegenerate, form.

Recall that for the matrices in the standard format $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with elements in a supercommutative superalgebra \mathcal{C} the *supertrace* is defined as

$$\text{str} : X = \text{tr} A - (-1)^{p(X)} \text{tr} D.$$

Let $\mathfrak{q}(n; \mathcal{C})$ be the Lie superalgebra of supermatrices $Y = \begin{pmatrix} A & B \\ (-1)^{p(Y)+1} B & (-1)^{p(Y)} A \end{pmatrix}$, where $A, B \in \mathfrak{gl}(p|q; \mathcal{C})$ with $p + q = n$. On it, str vanishes identically; instead, another

invariant function, the *queertrace* is defined:

$$\text{qtr}Y := \text{str} B.$$

(On matrices in the standard format, i.e., if $pq = 0$, the queertrace becomes $\text{qtr}Y := \text{tr} B$.)

For the matrix Lie superalgebras the standard (identity) representation provides with nondegenerate forms given by the formula

$$\begin{aligned} \langle x, y \rangle_{\text{id}} &= \text{str}(x \cdot y) \text{ is nondegenerate on } \mathfrak{gl}(m|n); \\ \langle x, y \rangle_{\text{id}} &= \text{qtr}(x \cdot y) \text{ is nondegenerate on } \mathfrak{q}(n) \end{aligned} \tag{1.1}$$

and the forms induced by the above forms on the quotient algebras \mathfrak{psl} and \mathfrak{psq} are also nondegenerate.

Under the contraction $\mathfrak{gl}(2^n|2^n) \longrightarrow \mathfrak{po}(0|2(n+1))$ and its restriction $\mathfrak{q}(2^n) \longrightarrow \mathfrak{po}(0|2n+1)$ (differential operator \mapsto its symbol; we consider the Poisson bracket on the symbols) the traces (1.1) turn into the integral and formulas (1.1) turn into

$$\langle x, y \rangle_{\text{id}} = \int (x \cdot y) \text{ vol} \text{ is nondegenerate on } \mathfrak{po}(0|N), \tag{1.2}$$

where *vol* is the volume element (for its (nontrivial in supersetting) definition see, e.g., [8]). The form (1.2) is even or odd together with N ; it induces a nondegenerate form on the simple Lie superalgebra $\mathfrak{sh}(0|N)$.

1.2 Bilinear forms on stringy superalgebras

Let \mathfrak{g} be the Lie superalgebra $\mathfrak{k}(\mathcal{M}^{2n+1|2n+6})$ of contact vector fields on \mathcal{M} , where we consider either a compact supermanifold \mathcal{M} or contact vector fields generated by functions with compact support. It so happens that for the dimension indicated $\mathfrak{k}(\mathcal{M}^{2n+1|2n+6})$ preserves the volume element *vol*, see [1]. Define the form B on \mathfrak{g} setting $B(K_f, K_g) = \int fg \text{ vol}$.

In particular, consider the Fourier images of the functions on the supercircle $S^{1|6}$; let $\mathfrak{k}^L(1|6)$ be the Lie superalgebra of the corresponding contact fields. It differs from $\mathfrak{k}(1|6)$ by the possibility to consider negative powers of $t = \exp i\varphi$, where φ is the angle parameter on S ; the superscript L indicates that we consider Laurent coefficients, rather than only polynomial ones. We set

$$B(K_f, K_g) = \text{Res} fg, \text{ where } \text{Res}(f) = \text{the coefficient of } \frac{\xi_1 \xi_2 \xi_3 \eta_1 \eta_2 \eta_3}{t}.$$

In [5] the even quadratic Casimir element Δ for $\mathfrak{k}^L(1|6)$ corresponding to B is explicitly computed. In terms of this element a solution of the classical Yang-Baxter equation can be expressed in the same way as in [9], namely, $\frac{\Delta}{u-v}$, and from this solution a quantum solution can be uniquely recovered thanks to the general uniqueness theorems of [2], [4].

In §2 we offer other solutions, without classical counterpart.

1.3 Open problems

It remains to explicitly produce the universal R -matrix for the triples of §2 (sec. 2.2–2.6) and classify the trigonometric solutions of the classical YB equation with values in $\mathfrak{k}^L(1|6)$ and in the Poisson superalgebras $\mathfrak{po}(0|2n)$, a problem left open in [9].

2 Main result: examples of Manin triples for Lie superalgebras

2.0. It is known that the extension of Drinfeld’s Example 3.2 in [3] to a simple Lie superalgebra \mathfrak{a} with a symmetrizable Cartan matrix (or, equivalently, an even supersymmetric nondegenerate form) has a new feature: there are *several* nonisomorphic Borel subsuperalgebras \mathfrak{a}^* ; but, nevertheless, $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}^*$ for all these Borel subalgebras.

(Observe that this implies that we can take for \mathfrak{a} nonisomorphic Borel subalgebras and get various types of defining relations for \mathfrak{g} . This observation applies not only to the superization of Example 3.2 in [3] — finite dimensional Lie superalgebras with a symmetrizable Cartan matrix — but to infinite dimensional superizations and the (analogues of) Borel subsuperalgebras of $\mathfrak{po}(0|2n)$ and $\mathfrak{k}^L(1|6)$ considered below. For a classification of systems of simple roots and for a description of Borel superalgebras see [11], for a description of defining relations see [6] and refs. therein.)

In their famous papers [7] Khoroshkin and Tolstoy *explicitly* wrote the universal R -matrices.

2.1. ([10]) $\mathfrak{a} = \mathfrak{q}(n)$; let $\mathfrak{a}^* = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \right\}$, where X is upper triangular, Z is lower triangular and $X_{ii} = -Z_{ii}$; then $\mathfrak{g} = \mathfrak{gl}(n|n)$ with the form $B(x, y) = \text{str}(xy)$. In this example the Lie superalgebra \mathfrak{a} is not simple. The construction can be transported to the simple algebra $\mathfrak{a} = \mathfrak{psq}(n)$ (cf. Remark 5 of [10]):

2.1’. $\mathfrak{a} = \mathfrak{psq}(n)$; let \mathfrak{a}^* be as in Example 2.1 but $\text{tr}X = \text{tr}Z = 0$; then $\mathfrak{g} = \mathfrak{psl}(n|n)$ with the form induced by $B(x, y) = \text{str}xy$. (In the last formula we consider the conditional presentation of the elements from $\mathfrak{psl}(n|n)$ by supermatrices from $\mathfrak{sl}(n|n)$ and the modified bracket: composition of the bracket with the subsequent subtraction of the supertrace.)

2.2. $\mathfrak{a} = \mathfrak{pe}(n)$; let $\mathfrak{a}^* = \left\{ \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \right\}$, where X, Z and T are upper triangular matrices and $X_{ii} = T_{ii}$, Y is a strictly upper triangular matrix; $\mathfrak{g} = \mathfrak{gl}(n|n)$ with the form $B(x, y) = \text{str}xy$.

2.2’. $\mathfrak{a} = \mathfrak{spe}(n) = \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n)$; let \mathfrak{a}^* be as in our Example 2.2 with $\text{tr}X = \text{tr}T = 0$; then $\mathfrak{g} = \mathfrak{psl}(n|n)$ with the form as in Example 2.1’.

2.3. $\mathfrak{a} = \mathfrak{po}(0|2n - 1)$. Let $\mathfrak{g} = \mathfrak{po}(0|2n)$ be generated by $\mathbb{C}[\theta_1, \dots, \theta_{2n}]$ and \mathfrak{a} by $\mathbb{C}[\theta_1, \dots, \theta_{2n-1}]$. The form B on \mathfrak{g} is $B(x, y) = \int xy \text{vol}(\theta)$. Let $\mathfrak{a}^* = \mathfrak{a} \cdot \zeta$, where $\zeta = \theta_{2n} + \sum_{1 \leq i \leq 2n-1} k_i \theta_i$ and $\sum_{1 \leq i \leq 2n-1} k_i^2 = -1$.

2.3’. $\mathfrak{a} = \mathfrak{psh}(2n - 1)$. Let $\mathfrak{g} = \mathfrak{psh}(0|2n)$ be conditionally (as in 2.1’) realized by generating functions $\mathbb{C}[\theta_1, \dots, \theta_{2n}]$ and \mathfrak{a} by $\mathbb{C}[\theta_1, \dots, \theta_{2n-1}]$. The form B on \mathfrak{g} is the one induced by the Berezin integral. Then $\mathfrak{a}^* = \mathfrak{a} \cdot \zeta$, with the ζ from 2.3.

2.4. $\mathfrak{g} = \mathfrak{po}(0|6)$. Let $\mathfrak{a} = \mathfrak{as} = \text{Span}(1, \Lambda(\xi, \eta), \Lambda^2(\xi, \eta), g_{11}^\xi)$, see [12]; then \mathfrak{a}^* is generated (as Lie superalgebra) by g_{11}^η (see [12]) and $\xi_1 \xi_2 \xi_3 \eta_1 \eta_2 \eta_3$.

2.5. Manin triples for $\mathfrak{g} = \mathfrak{k}^L(1|6)$. (Recall that superscript L indicates that we consider Laurent polynomials as generating functions, not just polynomials as in 2.5.1.)

2.5.1. \mathfrak{a} and \mathfrak{a}^* as in Drinfeld’s Example 3.3 [3], i.e., $\mathfrak{a} = \mathfrak{k}(1|6) \cong \text{Span}(K_f : f \in \mathbb{C}[t, \xi, \eta])$ and $\mathfrak{a}^* = \text{Span}(K_f : f \in t^{-1} \cdot \mathbb{C}[t^{-1}, \xi, \eta])$.

2.5.2. \mathfrak{a} and \mathfrak{a}^* as in Example 3.2 [3], for various Borel subalgebras.

2.5.3. A variation of our Example 2.3: $\mathfrak{a} = \mathfrak{k}^L(1|5)$ and $\mathfrak{a}^* = \mathfrak{a} \cdot \zeta$, where $\zeta = \theta_6 +$

$$\sum_{1 \leq i \leq 5} k_i \theta_i \text{ and } \sum_{1 \leq i \leq 5} k_i^2 = -1.$$

2.6. In what follows $V^1 = V \otimes \mathbb{C}[x^{-1}, x]$ for any vector space V . The form $B^{(1)}$ on $\mathfrak{g}^{(1)}$ is $\text{Res}_x B(f(x), g(x))$, where $x = \exp i\psi$ for the angle parameter ψ on T^1 and B is the nondegenerate invariant supersymmetric form on \mathfrak{g} . (Clearly, this T^1 and S from sec. 1.2 are diffeomorphic circles, as any two circles are. We denote them differently to underline that these are *different* circles.) Then new Manin triples $(\mathfrak{G}, \mathfrak{A}, \mathfrak{A}^*)$ are obtained from the triples $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$ corresponding to the examples 2.1–2.4 above for $\mathfrak{G} = \mathfrak{g}^{(1)}$, $\mathfrak{A} = \mathfrak{a}^{(1)}$, $\mathfrak{A}^* = (\mathfrak{a}^*)^{(1)}$.

Acknowledgements

We most gratefully acknowledge financial support of NFR and thank G. Olshansky for help.

References

- [1] Chaichian M., Leites D. and Lukierski E., New $N = 6$ Infinite Dimensional Superalgebra with Central Extension, *Phys. Lett.*, V.225, N 4, 1989, 347–351.
- [2] Chari, V. and Pressley, A., A Guide to Quantum Groups, Corrected reprint of the 1994 original, Cambridge University Press, Cambridge, 1995, xvi+651 pp.
- [3] Drinfeld V., Quantum Groups, Proc. ICM Berkeley 1986, AMS, 1987, V.1, 798–820.
- [4] Etingof P. and Kazhdan D., Quantization of Lie Bialgebras, I–V, q-alg 9506005, 9610030, 9701038, 9801043, 9808121; Quantization of Lie Bialgebras, I, *Selecta Math. (N.S.)*, 1996, V.2, N 1, 1–41; Quantization of Lie Bialgebras, II, III, *Selecta Math. (N.S.)*, 1998, V.4, N 2, 213–231, 233–269.
- [5] Grozman P. and Leites D., The Highest Weight Representations of the Contact Lie Superalgebra on 1|6-Dimensional Supercircle, *Czechoslovak J. Phys.*, 1997, V.47, N 11, 1133–1138.
- [6] Grozman P. and Leites D., Defining Relations for Lie Superalgebras with Cartan Matrix, hep-th 9702073.
- [7] Khoroshkin S. M. and Tolstoy V. N., Universal R -matrix for Quantum Supergroups, in Group theoretical methods in physics, (Moscow, 1990), Lecture Notes in Phys., 382, Springer, Berlin, 1991, 229–232; Universal R -Matrix for Quantized (Super)Algebras. *Comm. Math. Phys.*, 1991, V.141, N 3, 599–617; Extremal Projector and Universal R -Matrix for Quantized Contragredient Lie (Super)Algebras, in Quantum Groups and Related Topics, (Wrocław, 1991), Math. Phys. Stud., 13, Kluwer Acad. Publ., Dordrecht, 1992, 23–32; The Cartan-Weyl Basis and the Universal R -Matrix for Quantum Kac-Moody Algebras and Superalgebras, in Quantum Symmetries, (Clausthal, 1991), World Sci. Publishing, River Edge, NJ, 1993, 336–351.
- [8] Leites D., Quantization and Supermanifolds, In Schrödinger Equation, Editors F. Berezin and M. Shubin, Kluwer, Dordrecht, 1991.
- [9] Leites D. and Serganova V., The Solutions of the Classical Yang–Baxter Equation for Simple Lie Superalgebras, *Theor. Math. Phys.*, 1984, V.58, N 1, 26–37, (English translation in *Theoret. and Math. Phys.*, 1984, V.58, N 1, 16–24); Erratum and Addendum, in Seminar on Supermanifolds, Editor Leites D, Reports of Dept. of Math., University of Stockholm, Sweden, no. 23, 1988-5, 17–18.
- [10] Olshansky G., Quantized Universal Enveloping Superalgebra of Type Q and a Superextension of the Hecke Algebra, *Lett. Math. Phys.*, 1992, V.24, N 2, 93–102.

-
- [11] Penkov I. and Serganova V., Representations of Classical Lie Superalgebras of Type I, *Indag. Math. (N.S.)*, 1992, V.3, N 4, 419–466.
 - [12] Shchepochkina I., Five Exceptional Simple Lie Superalgebras, hep-th 9702121; Grozman P., Leites D. and Shchepochkina I., Lie Superalgebras of String Theories, hep-th/9702120.
 - [13] Shchepochkina I., Maximal Subalgebras of Matrix Lie Superalgebras, hep-th/9702122.