Correctors for the Homogenization of Monotone Parabolic Operators

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Abstract

In the homogenization of monotone parabolic partial differential equations with oscillations in both the space and time variables the gradients converges only weakly in $L^p$. In the present paper we construct a family of correctors, such that, up to a remainder which converges to zero strongly in $L^p$, we obtain strong convergence of the gradients in $L^p$.

1 Introduction

In [8] the asymptotic behaviour (as $\epsilon \to 0$) of the solutions $u_\epsilon$ to a sequence of initial-boundary value problems of the form

$$\begin{cases}
\frac{\partial u_\epsilon}{\partial t} - \text{div}(a(\frac{x}{\epsilon}, \frac{t}{\epsilon^\mu}, Du_\epsilon)) = f \text{ in } \Omega \times ]0, T[,
\end{cases}$$

$$u_\epsilon(x,0) = u_0(x),$$

$$u_\epsilon(x,t) = 0 \text{ in } \partial \Omega \times ]0, T[,$$

is studied. Here $\Omega$ is an open bounded set in $\mathbb{R}^N$, $T$ is a positive real number, $2 \leq p < \infty$ and $\mu > 0$. Under the assumption that $a(\frac{x}{\epsilon}, \frac{t}{\epsilon^\mu}, Du_\epsilon)$ is $\epsilon-$ and $\epsilon^\mu$-periodic in the first and second variable, respectively, it is proved that

$$u_\epsilon \to u \text{ weakly in } L^p(0, T; W^{1,p}_0(\Omega)),
$$

$$a(\frac{x}{\epsilon}, \frac{t}{\epsilon^\mu}, Du_\epsilon) \to b(Du) \text{ weakly in } L^{p'}(0, T; L^{p'}(\Omega; \mathbb{R}^N)),$$

where $1/p + 1/p' = 1$ and where $u$ denotes the unique solution to

$$\begin{cases}
\frac{\partial u}{\partial t} - \text{div}(b(Du)) = f \text{ in } \Omega \times ]0, T[,
\end{cases}$$

$$u(x,0) = u_0(x),$$

$$u(x,t) = 0 \text{ in } \partial \Omega \times ]0, T[,$$
where the limit map $b$ in (1.2) only depends on the sequence $(a(\cdot/\epsilon, \cdot/\epsilon^\mu, \xi))$ and on $\mu$ and where $b$ is different for $0 < \mu < 2$, $\mu = 2$ and $\mu > 2$, respectively. This result implies that

$$Du_\epsilon(x,t) = Du(x,t) + r_\epsilon(x,t),$$

where the remainder $r_\epsilon$ converges to zero only weakly in $L^p(0, T; L^p(\Omega; \mathbb{R}^N))$. The purpose of the present paper is to construct a family of correctors $(p_\epsilon) = (p_\epsilon(x, t, \xi))$ such that

$$p_\epsilon(\cdot, \cdot, \xi) \rightarrow \xi \text{ weakly in } L^p(0, T; L^p(\Omega; \mathbb{R}^N))$$

for every $\xi \in \mathbb{R}^N$ and

$$Du_\epsilon(x,t) = p_\epsilon(x, t, (M_\epsilon Du)(x,t)) + r_\epsilon(x,t),$$

where the remainder $r_\epsilon$ converges to zero strongly in $L^p(0, T; L^p(\Omega; \mathbb{R}^N))$ and where $(M_\epsilon)$ is a sequence of linear operators on $L^p(0, T; L^p(\Omega; \mathbb{R}^N))$ which converges strongly to the identity map on $L^p(0, T; L^p(\Omega; \mathbb{R}^N))$. The results presented in this article are rather technical and involves numerous estimates on small $\epsilon$-cubes. But the implications from Theorem 2.1 are important. In particular for computational modeling of (1.1) and (1.2) since it implies strong convergence of the gradients in the energy norm. In a simplified way we can say that the improvement of the convergence lies in the fact that the local behaviour on the $\epsilon$-cubes are added to the homogenized solution. Heuristically this amounts to adding the second term in an asymptotic expansion, see [1]. The corrector problem was first studied in [1] for linear elliptic and parabolic problems. For a careful study of linear parabolic problems we refer to [3]. See also [2] and [6]. The extension to the monotone elliptic case is performed in [4]. The present work is very much inspired by the methods developed in [4]. The paper is organized as follows: Section 2 contains some preliminaries and in Section 3 we present the main theorem (Theorem 3.1). In Section 4 we collect some useful estimates for the correctors and in Section 5 we give the proof of Theorem 3.1.

## 2 Preliminaries

Throughout this paper we will denote by $\Omega$ a bounded open set in $\mathbb{R}^N$ and we will let $V = W^{1,p}_0(\Omega)$, with norm $\|u\|_V^p = \int_\Omega |Du|^p \, dx$ and $V' = W^{-1,p'}(\Omega)$.

We consider the evolution triple

$$V \subseteq L^2(\Omega) \subseteq V',$$

with dense embeddings. Further, for positive real-valued $T$ and for $2 \leq p < \infty$, we define $\mathcal{V} = L^p(0, T; V)$ and $\mathcal{V}' = L^{p'}(0, T; V')$, where $1/p + 1/p' = 1$ and the corresponding evolution triple

$$\mathcal{V} \subseteq L^2([0, T] \times \Omega) \subseteq \mathcal{V}'$$

also with dense embeddings where the duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ between $\mathcal{V}$ and $\mathcal{V}'$ is given by

$$\langle f, u \rangle_{\mathcal{V}} = \int_0^T \langle f(t), u(t) \rangle_V \, dt, \text{ for } u \in \mathcal{V}, \ f \in \mathcal{V}'.$$
Given $u_0 \in L^2(\Omega)$, the space $W_0$ is defined as

$$W_0 = \{ v \in V : v' \in V' \text{ and } v(0) = u_0 \in L^2(\Omega) \}.$$ 

Here $v'$ denotes the time derivative of $v$, which is to be taken in distributional sense. Moreover, we define

$$U = L^p(0, T; L^p(\Omega; \mathbb{R}^N)) \text{ and } U' = L^{p'}(0, T; L^{p'}(\Omega; \mathbb{R}^N)).$$

with the duality pairing

$$\langle u, v \rangle_U = \int_0^T \int_{\Omega} (u, v) \, dx \, dt, \text{ for } u \in U' \text{ and } v \in U,$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^N$. By $|\cdot|$ we understand the usual Euclidean norm in $\mathbb{R}^N$ and by $m(\cdot)$ we understand the Lebesgue measure. Moreover, by $(e)$ we understand a sequence of positive real numbers tending to $0^+$.

Let $Y = [0, 1]^N$ be the unit cube in $\mathbb{R}^N$ and let $Y \times T_0 = [0, 1]^N \times [0, 1]$ be the unit cube in $\mathbb{R}^N \times \mathbb{R}_+^+$.

**Definition 2.1.** We say that a function $u : \mathbb{R}^N \times ]0, T[ \to \mathbb{R}$ is $Y$-periodic if $u(x + e_i, t) = u(x, t)$ for every $x \in \mathbb{R}^N$, $t \in ]0, T[$ and for every $i = 1, \ldots, N$, where $(e_i)$ is the canonical basis of $\mathbb{R}^N$. Further, we say that a function $u : \mathbb{R}^N \times \mathbb{R}_+^+ \to \mathbb{R}$ is $Y \times T_0$-periodic if $u(x + e_i, t) = u(x, t) = u(x, t + 1)$ for every $x \in \mathbb{R}^N$, $t \in \mathbb{R}_+$ and for every $i = 1, \ldots, N$.

We consider the following spaces of periodic functions:

$$V_{2,Y} = \{ u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) : u \text{ is } Y\text{-periodic and has mean value zero over } Y \},$$

and

$$V_{2,Y \times T_0} = \{ u \in L^p_{\text{loc}}(\mathbb{R}_+; V_{2,Y}) : u \text{ is } T_0 \text{-periodic} \}.$$

**Definition 2.2.** Given $0 < \alpha \leq 1$, $2 \leq p < \infty$ and three positive real constants $c_0$, $c_1$ and $c_2$ we define the class $S_{2,Y \times T_0} = S_{2,Y \times T_0}(c_0, c_1, c_2, \alpha)$ of maps

$$a : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N,$$

such that

(i) \hspace{1em} $a(\cdot, \cdot, \xi)$ is $Y \times T_0$-periodic for every $\xi \in \mathbb{R}^N$,

(ii) \hspace{1em} $|a(y, \tau, 0)| \leq c_0$ a.e in $\mathbb{R}^N \times \mathbb{R}_+$,

(iii) \hspace{1em} $a(\cdot, \cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$,

(iv) \hspace{1em} $|(a(y, \tau, \xi_1) - a(y, \tau, \xi_2)| \leq c_1 (1 + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1 - \xi_2|^\alpha$, a.e. in $\mathbb{R}^N \times \mathbb{R}_+$ for all $\xi_1, \xi_2 \in \mathbb{R}^N$,

(v) \hspace{1em} $(a(y, \tau, \xi_1) - a(y, \tau, \xi_2), \xi_1 - \xi_2) \geq c_2 |\xi_1 - \xi_2|^p$, a.e. in $\mathbb{R}^N \times \mathbb{R}_+$ for all $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$. 

We recall some results for maps $a \in S^\sharp_{\mathcal{Y} \times T_0}$:

**Proposition 2.1.** Suppose that $a \in S^\sharp_{\mathcal{Y} \times T_0}$. Then, for every $f \in \mathcal{V}'$ and for every $\epsilon > 0$, (1.1) possesses a unique solution $u_\epsilon \in W^0 \cap L^\infty(0,T;L^2(\Omega))$.

**Proof.** See e.g. [10].

**Proposition 2.2.** Let us put $a_\epsilon(\cdot,\cdot,\xi) = a(\frac{\cdot}{\epsilon},\frac{\cdot}{\epsilon^p},\xi)$. Suppose that $a \in S^\sharp_{\mathcal{Y} \times T_0}$. Then, for every $f \in \mathcal{V}'$, the solutions $u_\epsilon$ to (1.1) satisfy

$$u_\epsilon \to u \text{ weakly in } W^0,$$

$$a_\epsilon(x,t,Du_\epsilon) \to b(Du) \text{ weakly in } \mathcal{U'},$$

where $u$ is the unique solution to the following parabolic problem:

$$\begin{cases}
    u' - \text{div}(b(Du)) = f \text{ in } \Omega \times ]0,T[ \\
    u \in W^0.
  \end{cases}$$

Moreover, for a fixed vector $\xi \in \mathbb{R}^N$:

$$b(\xi) = \int_{\tau_0}^{\tau} \int_Y a(y,\tau,Dv(y,\tau) + \xi)\,dy\,d\tau,$$

where $v$ depends on $\xi$ and $\mu$. For $0 < \mu < 2$, $v = v(y,\tau)$ is the unique solution to the parameter-dependent elliptic problem:

$$\begin{cases}
    -\text{div}(a(y,\tau,Dv(y,\tau) + \xi)) = 0, \\
    v(\cdot,\tau) \in V^\sharp_{\mathcal{Y}}, \tau \geq 0.
  \end{cases}$$

For $\mu = 2$, $v = v(y,\tau)$ is the unique solution to the parabolic problem:

$$\begin{cases}
    v' - \text{div}(a(y,\tau,Dv(y,\tau) + \xi)) = 0, \\
    v \in V^\sharp_{\mathcal{Y} \times T_0}.
  \end{cases}$$

For $\mu > 2$, finally $v = v(y)$ is the unique solution to the elliptic problem:

$$\begin{cases}
    -\text{div}(\tilde{a}(y,Dv(y) + \xi)) = 0, \\
    v \in V^\sharp_{\mathcal{Y}},
  \end{cases}$$

where

$$\tilde{a}(y,\xi) = \int_{\tau_0}^{\tau} a(y,\tau,\xi)\,d\tau.$$
We close this section by stating some different Meyers type estimates which will be needed in the proof of the main corrector result, Theorem 3.1.

**Proposition 2.3.** Suppose that \( a \in S_{2,Y \times T_0} \). Let \( u \) be the solution to the problem
\[
\begin{cases}
-\text{div}(\tilde{a}(x, Du)) = 0, \\
u \in W^{1,p}(\Omega),
\end{cases}
\]
Then there exists a constant \( \eta > 0 \) such that \( u \in W^{1,p+\eta}(\tilde{\Omega}) \) for every open set \( \tilde{\Omega} \subset \subset \Omega \).

**Proof.** We refer to Theorem 1 in [7].

**Remark 2.** Considered as a function constant in \( t \geq 0 \), the function \( u \) above also satisfies the estimate
\[
\|u\|_{L^p(0,T;W^{1,p+\eta}(\tilde{\Omega}))} \leq C\|u\|_{L^p(0,T;W^{1,p}(\Omega))}.
\]

**Proposition 2.4.** Suppose that \( a \in S_{2,Y \times T_0} \) and in addition satisfies
\[
|a(x, t, \xi) - a(x, s, \xi)| \leq \omega(t - s)(1 + |\xi|^{p-1}) \quad (2.7)
\]
for all \( t, s \in [0,T] \), all \( \xi \in \mathbb{R}^N \) and a.e. \( x \in \Omega \), where \( \omega \) is the modulus of continuity. Let \( u(\cdot, t), t \in [0,T] \), be the solution to the parameter dependent elliptic problem
\[
\begin{cases}
-\text{div}(a(x, t, Du)) = 0, \\
u(\cdot, t) \in W^{1,p}(\Omega),
\end{cases}
\]
Then there exists a constant \( \eta > 0 \) such that, for every \( t \in [0,T] \), \( u(\cdot, t) \in W^{1,p+\eta}(\tilde{\Omega}) \) for every open set \( \tilde{\Omega} \subset \subset \Omega \). Moreover
\[
\|u(\cdot, t)\|_{W^{1,p+\eta}(\tilde{\Omega})} \leq C\|u(\cdot, t)\|_{W^{1,p}(\Omega)}.
\]

**Further, let \( \delta \subset [0,T] \). The gradient \( Du \) of the function \( u \) above also satisfies the estimate
\[
\|Du\|_{L^{p+\eta}(\delta;L^{p+\eta}(\tilde{\Omega};\mathbb{R}^N))} \leq C\|Du\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^N))}.
\]

**Proof.** The estimate (2.8) is a consequence of Theorem 1 in [7], if we take (2.7) into account. By using the coercivity of \( a \) and the Hölder inequality we get
\[
\begin{align*}
\int_{\delta} \|Du(\cdot, t + \epsilon) - Du(\cdot, t)\|_{L^p(\Omega)}^{p'} dt \\
& \leq c_2 \int_{\delta} \int_{\Omega} (a(x, t + \epsilon, Du(x, t + \epsilon)) - a(x, t, Du(x, t)), Du(x, t + \epsilon) - Du(x, t)) dx dt \\
& \leq c_2 \left( \int_{\delta} \|a(\cdot, t + \epsilon, Du(\cdot, t + \epsilon)) - a(\cdot, t, Du(\cdot, t))\|_{L^{p'}(\Omega)}^{p'} dt \right)^{1/p'} \\
& \quad \times \left( \int_{\delta} \|Du(\cdot, t + \epsilon) - Du(\cdot, t)\|_{L^p(\Omega)}^{p} dt \right)^{1/p}.
\end{align*}
\]
By (2.7) we obtain, using the Minkowski inequality and the boundedness of Ω,

\[
\left( \int_{\delta} \| Du(\cdot, t + \epsilon) - Du(\cdot, t) \|^p_{L^p(\Omega)} dt \right)^{\frac{1}{p'}} \leq c_2 \left( \int_{\delta} \| \omega(\epsilon) (1 + |\max\{Du(\cdot, t + \epsilon), Du(\cdot, t)\}|) \|_{L^p(\Omega)} \right)^{1/p'} \\
\leq c_2 \left( \int_{\delta} |\omega(\epsilon)| L^{p'}(\Omega) + \| \max\{Du(\cdot, t + \epsilon), Du(\cdot, t)\}\|_{L^p(\Omega)} \right)^{1/p'}.
\]

For \( \epsilon \) small enough \( \omega(\epsilon) \leq 1 \) and (2.8) implies that

\[
\sup_{t \in \delta} \| \max\{Du(\cdot, t + \epsilon), Du(\cdot, t)\}\|_{L^p(\Omega)} \leq C,
\]

where \( C \) is independent of \( \epsilon \). Therefore

\[
\int_{\delta} \| Du(\cdot, t + \epsilon) - Du(\cdot, t) \|^p_{L^p(\Omega)} dt \leq C \int_{\delta} |\omega(\epsilon)| dt,
\]

which tends to zero as \( \epsilon \to 0 \), by the dominated convergence theorem. The estimate (2.9) now readily follows by the continuity of \( Du \) with respect to \( t \).

\[\blacksquare\]

**Proposition 2.5.** Suppose that \( a \in S_{p,Y \times T_0} \). Let \( u \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \) be the solution to the problem

\[
u' - \text{div}(a(x, t, Du) = 0.
\]

Let \( \tilde{\Omega} \) be defined as above and let \( \delta \subset\subset [0, T] \). The gradient \( Du \) of the function \( u \) above also satisfies the estimate

\[
\| Du \|_{L^q(\delta; L^q(\tilde{\Omega}; \mathbb{R}^N))} \leq C \| Du \|_{L^p(0, T; L^p(\Omega; \mathbb{R}^N))}.
\]

for any \( q \in [1, \infty] \).

**Proof.** We refer to Lemma 2.2 and Remark 7.4 of [5].

### 3 The main result

In this section we state the main corrector result which we indicated in the previous sections. We start out by defining a sequence \( (M_\epsilon) \) of approximations of the identity map on \( \mathcal{U} \). For \( i \in \mathbb{Z}^N \) and \( j \in \mathbb{Z} \) we consider the translated images \( Y^i_\epsilon = \epsilon(i + Y) \) and \( T^{j}_0,\epsilon = \epsilon(j + T_0) \). Take \( \varphi \in \mathcal{U} \). We define the function

\[
M_\epsilon \varphi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N
\]

by

\[
(M_\epsilon \varphi)(x, t) = \sum_{i \in I_\epsilon} \sum_{j \in I_\epsilon} \chi_{Y^i_\epsilon}(x) \chi_{T^{j}_0,\epsilon}(t) \frac{1}{m(Y^i_\epsilon \times T^{j}_0,\epsilon)} \int_{T^{j}_0,\epsilon} \int_{Y^i_\epsilon} \varphi(y, \tau) dy d\tau,
\]

(3.1)
where
\[ I_\epsilon = \{ i \in \mathbb{Z}^N : Y^i_\epsilon \in \Omega \} \text{ and } J_\epsilon = \{ j \in \mathbb{Z} : T_{0,\epsilon}^j \in [0, T] \} \]
and \( \chi_A \) denotes the characteristic function of the measurable set \( A \). It is well-known that
\[ M_\epsilon \varphi \rightharpoonup \varphi \text{ strongly in } \mathcal{U}. \quad (3.2) \]
By the Jensen’s inequality we also have
\[ \| M_\epsilon \varphi \|_U \leq \| \varphi \|_U \quad (3.3) \]
for all \( \varphi \in \mathcal{U} \).

Let us also define the \( Y \times T_0 \)-periodic function
\[ p : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, \]
which depends on \( \mu \), by
\[ p(x, t, \xi) = \xi + Dv(x, t), \quad (3.4) \]
where \( v \) is the solution to the auxiliary local problem (2.3), (2.4) or (2.5) for \( 0 < \mu < 2 \), \( \mu = 2 \) and \( \mu > 2 \), respectively. It follows that the function
\[ p_\epsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \]
defined by
\[ p_\epsilon(x, t, \xi) = \xi + Dv(\frac{x}{\epsilon}, \frac{t}{\epsilon^\mu}), \quad (3.5) \]
is \( \epsilon Y \)-periodic in \( x \) and \( \epsilon^\mu T_0 \)-periodic in \( t \). This means that
\[ \int_{T_0} \int_Y p(x, t, \xi) \, dxdt = \xi \]
and that
\[ p_\epsilon(\cdot, \cdot, \xi) \rightharpoonup \xi \text{ weakly in } \mathcal{U}. \quad (3.6) \]
Thus, the homogenized map \( b \) can be expressed as
\[ b(\xi) = \int_{T_0} \int_Y a(x, t, p(x, t, \xi)) \, dxdt. \quad (3.7) \]
Moreover, we have
\[ \int_{T_0} \int_Y (a(x, t, p(x, t, \xi)), p(x, t, \xi)) \, dxdt = \int_{T_0} \int_Y (a(x, t, p(x, t, \xi)), \xi) \, dxdt = (b(\xi), \xi). \quad (3.8) \]
The following correctors result is the main result of this paper:
Theorem 3.1. Suppose that \( a \in S_{Y,T_0} \). For the case \( 0 < \mu < 2 \) we also suppose that \( a \) satisfies (2.7). Moreover, suppose that \( f \in \mathcal{V}' \) and let \( u_\epsilon \) be the solutions to (1.1) and let \( u \) be the solution to (1.2). Then, we have

\[
Du_\epsilon = p_\epsilon(\cdot,\cdot,M_\epsilon Du) + r_\epsilon,
\]

where \( p_\epsilon \) is defined by (3.5) and where

\[
r_\epsilon \to 0 \quad \text{strongly in } U.
\]

Remark 3. Recall that \( p_\epsilon \) is entirely different for the three cases \( 0 < \mu < 2 \), \( \mu = 2 \) and \( \mu > 2 \), respectively.

4 Some estimates for the family of correctors

In this section we present some estimates for the family \((p_\epsilon)\) of correctors. To a large extent the proofs will follow by minor modifications of the proofs of similar lemmas by Dal Maso and Defranceschi in [4]. Therefore we refer to their paper for complete details and present here only proofs of parts which require more modifications.

Lemma 4.1. For any vector \( \xi \in \mathbb{R}^N \) we have

\[
\|p_\epsilon(\cdot,\cdot,\xi)\|_{L^p(T_{0,\epsilon};L^p(Y_\epsilon;\mathbb{R}^N))} \leq C(1 + |\xi|^p)m(Y_\epsilon \times T_{0,\epsilon}),
\]

where the constant \( C \) depends only on \( N, p, c_0, c_1 \) and \( c_2 \).

Lemma 4.2. There exist \( \eta > 0 \) and \( C > 0 \), which depends only on \( N, p, c_0, c_1 \) and \( c_2 \), such that

\[
\|p_\epsilon(\cdot,\cdot,\xi)-p_\epsilon(\cdot,\cdot,\xi)\|_{L^{p+\eta}(T_{0,\epsilon};L^{p+\eta}(Y_\epsilon;\mathbb{R}^N))} \leq C(1 + |\xi|^{p+\eta})m(Y_\epsilon \times T_{0,\epsilon}),
\]

for every \( \xi \in \mathbb{R}^N \).

Proof. By referring to the Meyers estimates in Propositions 2.3, 2.4 and 2.5 the proof is analogous as the proof of Corollary 3.3 in [4].

Lemma 4.3. For every \( \xi_1, \xi_2 \) in \( \mathbb{R}^N \) we have

\[
\|p_\epsilon(\cdot,\cdot,\xi_1) - p_\epsilon(\cdot,\cdot,\xi_2)\|_{L^p(T_{0,\epsilon};L^p(Y_\epsilon;\mathbb{R}^N))} \\
\leq C(1 + |\xi_1|^p + |\xi_2|^p)^{(p-1-\alpha)/(p-\alpha)}|\xi_1 - \xi_2|^{p/(p-\alpha)}m(Y_\epsilon \times T_{0,\epsilon}),
\]

where the constant \( C \) depends only on \( N, p, \alpha, c_0, c_1 \) and \( c_2 \).

Lemma 4.4. Let \( \varphi \in \mathcal{U} \) and consider a simple function \( \Psi \) given by

\[
\Psi(x,t) = \sum_{k=1}^m c_k \chi_{\Omega_k}(x) \chi_{\delta_k}(t),
\]
with \( c_k \in \mathbb{R}^N \setminus \{0\}, \Omega_k \subset \subset \Omega, \delta_k \subset \subset ]0, T[, m(\partial \Omega_k) = m(\partial \delta_k) = 0 \) and \((\Omega_k \cap \Omega_l) \times (\delta_k \cap \delta_l) = \phi \) for \( k \neq l \). Then

\[
\begin{align*}
\limsup_{\epsilon \to 0} & \| p_\epsilon(\cdot \cdot, M_\epsilon \varphi) - p_\epsilon(\cdot \cdot, \Psi) \|_U \\
& \leq C(m(\Omega \times ]0, T[) + \| \varphi \|_U + \| \Psi \|_U)^{(p-1-\alpha)/(p-\alpha)} \| \varphi - \Psi \|_U^{1/(p-\alpha)},
\end{align*}
\]

(4.4)

where \( C \) depends only on \( N, p, \alpha, c_0, c_1 \) and \( c_2 \).

**Proof.** Put \( \Omega_0 = \Omega \setminus \cup_{k=1}^m \Omega_k, \delta_0 = ]0, T[ \setminus \cup_{k=1}^m \delta_k \) and \( c_0 = 0 \). Then we have

\[
\Psi(x, t) = \sum_{k=0}^m c_k \chi_{\Omega_k}(x) \chi_{\delta_k}(t).
\]

For every \( \epsilon > 0 \) we denote by \( \Omega_\epsilon \times \delta_\epsilon \) the union of all closed cubes \( \Gamma_\epsilon^i \times T_0^j_\epsilon \) such that \( Y_\epsilon^i \subset \Omega \) and \( T_{0, \epsilon}^j \subset ]0, T[. \) For \( k = 0, 1, \ldots, m \) we define the sets

\[
I_\epsilon^k = \{ i \in I_\epsilon : Y_\epsilon^i \subset \Omega_k \}, \quad J_\epsilon^k = \{ j \in J_\epsilon : T_{0, \epsilon}^j \subset \delta_k \},
\]

and

\[
\tilde{I}_\epsilon^k = \{ i \in I_\epsilon : Y_\epsilon^i \cap \Omega_k \neq \phi \}, \quad \tilde{J}_\epsilon^k = \{ j \in J_\epsilon : T_{0, \epsilon}^j \cap \delta_k \neq \phi \}.
\]

Further, we define \( E_{\epsilon, j, k} \) as the union of all closed cubes \( \Gamma_\epsilon^i \times T_{0, \epsilon}^j \) with \( i \in I_\epsilon^k \) and \( j \in J_\epsilon^k \), and we define \( \tilde{E}_{\epsilon, j, k} \) as the union of all closed cubes \( \Gamma_\epsilon^i \times T_{0, \epsilon}^j \) with \( i \in \tilde{I}_\epsilon^k \) and \( j \in \tilde{J}_\epsilon^k \). If we choose \( \epsilon \) small enough, then, for \( k \neq 0 \), \( \Omega_k \times \delta_k \subset \Omega_\epsilon \times \delta_\epsilon \) according to (3.1). Thus, the definition of \( \Psi \) yields

\[
\begin{align*}
\| p_\epsilon(\cdot \cdot, M_\epsilon \varphi) - p_\epsilon(\cdot \cdot, \Psi) \|_U^p & = \int_{\delta_\epsilon} \int_{\Omega_\epsilon} | p_\epsilon(x, t, M_\epsilon \varphi) - p_\epsilon(x, t, \Psi) |^p \, dx \, dt \\
& \leq \sum_{k=0}^m \int_{E_{\epsilon, j, k}} | p_\epsilon(x, t, M_\epsilon \varphi) - p_\epsilon(x, t, c_k) |^p \, dx \, dt \\
& \quad + \sum_{k=0}^m \int_{\tilde{E}_{\epsilon, j, k}} | p_\epsilon(x, t, M_\epsilon \varphi) - p_\epsilon(x, t, c_k) |^p \, dx \, dt.
\end{align*}
\]

Let us put

\[
\theta_{\epsilon}^{i,j} = \frac{1}{m(Y_\epsilon^i \times T_{0, \epsilon}^j)} \int_{T_{0, \epsilon}^j} \int_{Y_\epsilon^i} \varphi(x, t) \, dx \, dt.
\]

A repeated application of the Hölder’s and the Jensen’s inequalities yields, according to Lemma 4.3,

\[
\| p_\epsilon(\cdot \cdot, M_\epsilon \varphi) - p_\epsilon(\cdot \cdot, \Psi) \|_U^p
\]
\[
\leq C \sum_{k=0}^{m} \left( \sum_{j \in \mathcal{J}_k} \sum_{i \in \mathcal{I}_k} (1 + |\theta_{ij}^{i,j}|^p + |c_k|^p (p-1-\alpha)/(p-\alpha)) |\theta_{ij}^{i,j} - c_k|^p/(p-\alpha) m(Y_i^j \times T_0^j) \right)
\]  

+ \[
C \sum_{k=0}^{m} \left( \sum_{j \in \mathcal{J}_k} \sum_{i \in \mathcal{I}_k} (1 + |\theta_{ij}^{i,j}|^p + |c_k|^p (p-1-\alpha)/(p-\alpha)) |\theta_{ij}^{i,j} - c_k|^p/(p-\alpha) m(Y_i^j \times T_0^j) \right)
\]

\[
\leq C \sum_{k=0}^{m} \left( m(E_{ij}^{i,j,k}) + \int_{E_{ij}^{i,j,k}} |\varphi|^p dxdt + |c_k|^p m(E_{ij}^{i,j,k}) \right)^{\frac{p-1-\alpha}{p-\alpha}} \|\varphi - \Psi\|^p/(p-\alpha)
\]

+ \[
C \sum_{k=0}^{m} \left( m(E_{ij}^{i,j,k}) + \int_{E_{ij}^{i,j,k}} |\varphi|^p dxdt + |c_k|^p m(E_{ij}^{i,j,k}) \right)^{\frac{p-1-\alpha}{p-\alpha}} \|\varphi - c_k\|^p/(p-\alpha)
\]

Hence

\[
\|p_\epsilon(\cdot, \cdot, M_\epsilon \varphi) - p_\epsilon(\cdot, \cdot, \Psi)\|_U^p \leq C(m(\Omega \times ]0, T[) + \|\varphi\|^p_U + \|\Psi\|^p_U) (p-1-\alpha)/(p-\alpha) \|\varphi - \Psi\|^p/(p-\alpha)
\]

\[
+ C \sum_{k=0}^{m} \left( m(E_{ij}^{i,j,k}) + \int_{E_{ij}^{i,j,k}} |\varphi|^p dxdt + |c_k|^p m(E_{ij}^{i,j,k}) \right)^{\frac{p-1-\alpha}{p-\alpha}} \|\varphi - c_k\|^p/(p-\alpha)
\]

(4.5)

Now recall that \(m(\partial \Omega_k) = m(\partial \delta_k) = 0\) for \(k \neq 0\). Thus, \(m(E_{ij}^{i,j,k}) \to 0\) as \(\epsilon \to 0\) for every \(k = 0, 1, \ldots, m\) and the lemma is proved.

\section{Proof of the main corrector result}

In this section we give the proof of the main corrector result, Theorem 3.1, stated in Section 3. Our proof will follow the lines of the proof of the corrector result for the corresponding elliptic problem, earlier proved by Dal Maso and Defranceschi in [4]. We start out by proving an estimate on \(p_\epsilon(\cdot, \cdot, M_\epsilon Du)\) uniformly with respect to \(\epsilon\).

\begin{lemma}
Let \(p_\epsilon\) be defined as in (3.5). Then,

\[
\|p_\epsilon(\cdot, \cdot, M_\epsilon Du)\|_U^p \leq C,
\]

where the positive constant \(C\) is independent of \(\epsilon\).
\end{lemma}

\begin{proof}
Let us define

\[
\theta_{ij}^{i,j} = \frac{1}{m(Y_i^j \times T_0^j) \int_{T_0^j} \int_{Y_i^j} Du(x, t) dx dt}.
\]

and

\[
\bar{I}_\epsilon = \{ i \in \mathbb{Z}^N : Y_i^j \cap \Omega \neq \phi, Y_i^j \setminus \Omega \neq \phi \},
\]

\[
\bar{J}_\epsilon = \{ j \in \mathbb{Z} : T_0^j \cap ]0, T[ \neq \phi, T_0^j \setminus ]0, T[ \neq \phi \}.
\]

\end{proof}
We apply Lemma 4.1, Lemma 4.2 and the inequality (3.3) to obtain
\[
\|p_\epsilon(\cdot, \cdot, M_\epsilon D u)\|_H^p \\
= \sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} \int_{T_0 \times Y_\epsilon^i} |p_\epsilon(x, t, \theta^{i,j}_\epsilon)|^p \, dx \, dt + \int_{[0,T] \times \Omega} |p_\epsilon(x, t, 0)|^p \, dx \, dt \\
\leq \sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} C(1 + |\theta^{i,j}_\epsilon|^p)m(Y_\epsilon^i \times T_{0,\epsilon}^j) \\
+ \left( \sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} \|p_\epsilon(\cdot, \cdot, 0)\|_{L^{p+\eta}([0,T] \times \Omega; \mathbb{R}^N)}^{p+\eta} \right)^{p/(p+\eta)} \\
\times m((\Omega \times [0,T]) \times ([0,T] \times \delta_\epsilon))^{\eta/(p+\eta)} \\
\leq Cm((\Omega \times [0,T]) + C\|M_\epsilon D u\|_H^p + C \left( \sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} m(Y_\epsilon^i \times T_{0,\epsilon}^j) \right)^{p/(p+\eta)} \\
\times m((\Omega \times [0,T]) \times ([0,T] \times \delta_\epsilon))^{\eta/(p+\eta)} \\
\leq Cm((\Omega \times [0,T]) + C\|D u\|_H^p + C \left( \sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} m(Y_\epsilon^i \times T_{0,\epsilon}^j) \right)^{p/(p+\eta)} \\
\times m((\Omega \times [0,T]) \times ([0,T] \times \delta_\epsilon))^{\eta/(p+\eta)}. \quad (5.2)
\]

Now \( \sum_{j \in J_\epsilon} \sum_{i \in I_\epsilon} m(Y_\epsilon^i \times T_{0,\epsilon}^j) \) approaches \( m(\partial\Omega \times \partial[0,T]) \) and \( m((\Omega \times \delta_\epsilon) \times ([0,T] \times \delta_\epsilon)) \) tends to zero as \( \epsilon \to 0 \). Thus, (5.1) follows by (5.2) and Lemma 5.1 is proved.

**Proof of Theorem 3.1.** By the strict monotonicity assumption it follows that
\[
\|p_\epsilon(\cdot, \cdot, M_\epsilon D u) - D u_\epsilon\|_H \\
\leq C \left( \int_0^T \int_{\Omega} (a_\epsilon(x, t, p_\epsilon(x, t, M_\epsilon D u)) - a_\epsilon(x, t, D u_\epsilon), p_\epsilon(x, t, M_\epsilon D u) - D u_\epsilon) \, dx \, dt \right)^{1/p}. \quad (5.3)
\]

Consequently, Theorem 3.1 is proved if we can prove that
\[
\int_0^T \int_{\Omega} (a_\epsilon(x, t, p_\epsilon(x, t, M_\epsilon D u)) - a_\epsilon(x, t, D u_\epsilon), D u_\epsilon) \, dx \, dt \to 0 \quad (5.4)
\]
as \( \epsilon \to 0 \). The proof of (5.4) will be splitted up into four steps.

**Step 1.** We start by showing that
\[
\int_0^T \int_{\Omega} (a_\epsilon(x, t, p_\epsilon(x, t, M_\epsilon D u), p_\epsilon(x, t, M_\epsilon D u)) \, dx \, dt \to \int_0^T \int_{\Omega} (b(D u), D u) \, dx \, dt. \quad (5.5)
\]

Let us write
\[
\int_0^T \int_{\Omega} (a_\epsilon(x, t, p_\epsilon(x, t, M_\epsilon D u), p_\epsilon(x, t, M_\epsilon D u)) \, dx \, dt
\]
By arguing as in Lemma 5.1 we conclude that

\[ \sum_{j \in J, i \in I_c} \int_{T_0^{j,i}} \int_{Y_{j,i}} \left( a \left( \frac{x}{\epsilon}, \frac{t}{\epsilon}, p \left( \frac{x}{\epsilon}, \frac{t}{\epsilon}, \theta_{\epsilon} \right) \right), p \left( \frac{x}{\epsilon}, \frac{t}{\epsilon}, \theta_{\epsilon} \right) \right) dx \] 

\[ + \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (a \left( x, t, p \left( x, t, 0 \right) \right), p \left( x, t, 0 \right) ) dx dt \]

\[ = \epsilon^{N+1} \sum_{j \in J, i \in I_c} \int_{T_0^{j,i}} \int_{Y} (a \left( y, \tau, p \left( y, \tau, \theta_{\epsilon} \right) \right), p \left( y, \tau, \theta_{\epsilon} \right) ) dy d\tau \]

\[ + \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (a \left( x, t, p \left( x, t, 0 \right) \right), p \left( x, t, 0 \right) ) dx dt \]

\[ = \sum_{j \in J, i \in I_c} \int_{0}^{T} \int_{\Omega} \chi_{Y_j} (y) \chi_{T_0^{j,i}} (t) (b \left( \theta_{\epsilon}^{i,j}, \theta_{\epsilon} \right) ) dy d\tau \]

\[ + \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (a \left( x, t, p \left( x, t, 0 \right) \right), p \left( x, t, 0 \right) ) dx dt, \quad (5.6) \]

where the last equality follows from (3.8). According to Remark 1 the map \( \varphi \rightarrow b(\varphi) \) is continuous from \( \mathcal{U} \) into \( \mathcal{U}' \) and an application of (3.2), using this fact, yields

\[ b(M_c Du) \rightarrow b(Du) \quad \text{strongly in} \quad \mathcal{U}'. \quad (5.7) \]

and, thus,

\[ \sum_{j \in J, i \in I_c} \int_{0}^{T} \int_{\Omega} \chi_{Y_j} (y) \chi_{T_0^{j,i}} (t) (b \left( \theta_{\epsilon}^{i,j}, \theta_{\epsilon} \right) ) dy d\tau \]

\[ = \int_{0}^{T} \int_{\Omega} (b(M_c Du, M_c Du) dy d\tau \rightarrow \int_{0}^{T} \int_{\Omega} (b(Du, Du) dy d\tau. \quad (5.8) \]

By the uniform continuity assumption we have

\[ | \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (a \left( x, t, p \left( x, t, 0 \right) \right), p \left( x, t, 0 \right) ) dx dt | \]

\[ \leq C \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (1 + |p \left( x, t, 0 \right) )|p dx dt \]

\[ + | \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (a \left( x, t, 0 \right), p \left( x, t, 0 \right) ) dx dt | \]

\[ \leq C m((\Omega \setminus \Omega_{\delta}) \times ([0,T] \setminus \delta)) + C \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} |p \left( x, t, 0 \right) |^p dx dt \]

\[ + C \left( \sum_{j \in J, i \in I_c} m(Y_i \times T_{0,i}^{j}) \right)^{1/p'} \left( \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} |p \left( x, t, 0 \right) |^p dx dt \right)^{1/p}. \]

By arguing as in Lemma 5.1 we conclude that

\[ | \int_{[0,T] \setminus \delta} \int_{\Omega \setminus \Omega_{\delta}} (a \left( x, t, p \left( x, t, 0 \right) \right), p \left( x, t, 0 \right) ) dx dt | \rightarrow 0. \]
Thus, by taking (5.6) and (5.8) into account we have shown (5.5).

**Step 2.** We proceed by showing that

\[
\int_0^T \int_{\Omega} (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)), Du_\epsilon) \, dx dt \rightarrow \int_0^T \int_{\Omega} (b(Du), Du) \, dx dt.
\]

Let \(\rho > 0\) be arbitrary. For \(Du \in \mathcal{U}\) there exists a simple function

\[
\Psi = \sum_{k=1}^{m} c_k \chi_{\Omega_k} \chi_{\delta_k},
\]

which satisfies the assumptions in Lemma 4.4, such that

\[
\|Du - \Psi\|_{\mathcal{U}} \leq \rho.
\]

We write

\[
\int_0^T \int_{\Omega} (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dx dt = \int_0^T \int_{\Omega} (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dx dt + \int_0^T \int_{\Omega} (a_\epsilon(x,t,p_\epsilon(x,t,M_\epsilon Du)) - (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dx dt.
\]

It follows, for the first integral on the right hand side, that

\[
\int_0^T \int_{\Omega} (a_\epsilon(x,t,p_\epsilon(x,t,\Psi)), Du_\epsilon) \, dx dt = \sum_{k=0}^{m} \int_{\delta_k} \int_{\Omega_k} (a_\epsilon(x,t,p_\epsilon(x,t,c_k)), Du_\epsilon) \, dx dt,
\]

where \(c_0 = 0\) and where \(\Omega_0\) and \(\delta_0\) are defined as in the previous section. By Lemma 4.2, the functions \(p_\epsilon(\cdot,\cdot,c_k)\) are bounded in \(L^{p+\eta}(0,T;L^{p+\eta}(\Omega;\mathbb{R}^N))\). By the structure conditions this implies that \(a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k))\) is uniformly bounded in \(L^s(0,T;L^s(\Omega;\mathbb{R}^N))\) for some \(s > p'\). From Proposition 2.2 it further follows that the sequence \((Du_\epsilon)\) is bounded in \(\mathcal{U}\). Therefore there exists a number \(\sigma > 1\) such that

\[
\|(a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)), Du_\epsilon)\|_{L^\sigma([0,T] \times \Omega)} \leq C
\]

uniformly with respect to \(\epsilon\). Hence, up to a subsequence,

\[
(a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)), Du_\epsilon) \rightarrow g_k \text{ weakly in } L^\sigma([0,T] \times \Omega),
\]

as \(\epsilon \rightarrow 0\). By proposition 2.2 we know that

\[
a_\epsilon(\cdot,\cdot,p_\epsilon(\cdot,\cdot,c_k)) \rightarrow b(c_k) \text{ weakly in } \mathcal{U}'.
\]
This enables us to use the compensated compactness result Theorem 2.1 in [9] and conclude that

\[(a_\varepsilon(\cdot, \cdot, p_\varepsilon(\cdot, c_k)), Du_\varepsilon) \to (b(c_k), Du)\]

in the sense of distributions. Consequently \(g_k = (b(c_k), Du)\) and

\[
\sum_{k=0}^{m} \int_{\Omega_k} (a_\varepsilon(x, t, p_\varepsilon(x, t, c_k)), Du_\varepsilon) \, dxdt \to \sum_{k=0}^{m} \int_{\Omega_k} (b(c_k), Du) \, dxdt.
\]

By using (5.12) this gives

\[
\int_0^T \int_{\Omega} (a_\varepsilon(x, t, p_\varepsilon(x, t, \Psi)), Du_\varepsilon) \, dxdt \to \int_0^T \int_{\Omega} (b(\Psi), Du) \, dxdt. \tag{5.13}
\]

For the second integral on the right hand side of (5.11) we observe that the growth condition on \(a_\varepsilon\) together with the Hölder inequality gives

\[
\left| \int_0^T \int_{\Omega} (a_\varepsilon(x, t, p_\varepsilon(x, t, M_\varepsilon Du)) - (a_\varepsilon(x, t, p_\varepsilon(x, t, \Psi))), Du_\varepsilon) \, dxdt \right|
\]

\[
\leq C \int_0^T \int_{\Omega} (1 + |p_\varepsilon(x, t, M_\varepsilon Du)|^p + |p_\varepsilon(x, t, \Psi)|^p)^{(p-1-\alpha)/p} \times |p_\varepsilon(x, t, M_\varepsilon Du) - p_\varepsilon(x, t, \Psi)|^\alpha |Du_\varepsilon| \, dxdt
\]

\[
\leq C \left( \int_0^T \int_{\Omega} (1 + |p_\varepsilon(x, t, M_\varepsilon Du)|^p + |p_\varepsilon(x, t, \Psi)|^p)^{(p-1-\alpha)/p} \right)
\]

\[
\times \left( \int_0^T \int_{\Omega} |Du_\varepsilon|^p \, dxdt \right)^{1/p} |p_\varepsilon(x, t, M_\varepsilon Du) - p_\varepsilon(x, t, \Psi)|^p \, dxdt)^{\alpha/p}. \tag{5.14}
\]

By the Lemmas 5.1 and 4.1 the sequences \((p_\varepsilon(\cdot, \cdot, M_\varepsilon Du))\) and \((p_\varepsilon(\cdot, \cdot, \Psi))\) are bounded in \(U\). Therefore, by using (5.10), the last inequality in (5.14) and Lemma 4.4 gives

\[
\limsup_{\varepsilon \to 0} \left| \int_0^T \int_{\Omega} (a_\varepsilon(x, t, p_\varepsilon(x, t, M_\varepsilon Du)) - (a_\varepsilon(x, t, p_\varepsilon(x, t, \Psi))), Du_\varepsilon) \, dxdt \right| \leq C \rho^\gamma. \tag{5.15}
\]

By taking Remark 1 into account we obtain

\[
\left| \int_0^T \int_{\Omega} (b(Du) - b(\Psi), Du) \, dxdt \right|
\]

\[
\leq C \int_0^T \int_{\Omega} (1 + |Du|^p + |\psi|^p)^{(p-1-\gamma)/p} |Du - \Psi|^\gamma |Du| \, dxdt.
\]

Again using the Hölder inequality, and (5.10), yields

\[
\left| \int_0^T \int_{\Omega} (b(Du) - b(\Psi), Du) \, dxdt \right|
\]

\[
\leq C \left( \int_0^T \int_{\Omega} (1 + |Du|^p + |\psi|^p) \, dxdt \right)^{(p-1-\gamma)/p} \left( \int_0^T \int_{\Omega} |Du|^p \, dxdt \right)^{1/p} \rho^\gamma \leq C \rho^\gamma. \tag{5.16}
\]
Thus (5.9) follows by the arbitrariness of $\rho$ and Step 2 is accomplished.

**Step 3** We show that
\[
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), p_\epsilon(x,t,M_\epsilon Du)) \, dx dt \to \int_0^T \int_\Omega (b(Du), Du) \, dx dt. \tag{5.17}
\]
Let us fix $\delta > 0$ and let $\Psi$ be defined as in Step 2. We write
\[
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), p_\epsilon(x,t,M_\epsilon Du)) \, dx dt = \sum_{k=0}^m \int_{\delta_k} \int_{\Omega_k} (a_\epsilon(x,t,Du_\epsilon(x,t,c_k)), p_\epsilon(x,t,c_k)) \, dx dt + \int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)) \, dx dt. \tag{5.18}
\]
By similar arguments as in Step 2 we conclude that
\[
\sum_{k=0}^m \int_{\delta_k} \int_{\Omega_k} (a_\epsilon(x,t,Du_\epsilon(x,t,c_k)), p_\epsilon(x,t,c_k)) \, dx dt \to \int_0^T \int_\Omega (b(Du), \Psi) \, dx dt. \tag{5.19}
\]
It also follows, by the Hölder inequality, that
\[
| \int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)) \, dx dt | \leq ( \int_0^T \int_\Omega |a_\epsilon(x,t,Du_\epsilon)|^{p'} \, dx dt )^{1/p'} \times ( \int_0^T \int_\Omega |p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)|^p \, dx dt )^{1/p}.
\]
Therefore, according to Lemma 4.4,
\[
\limsup_{\epsilon \to 0} | \int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), p_\epsilon(x,t,M_\epsilon Du) - p_\epsilon(x,t,\Psi)) \, dx dt | \leq C \rho^{1/p - \alpha}. \tag{5.20}
\]
(5.17) now follows by an analogous argumentation as in the final lines of Step 2.

**Step 4** In order to conclude the proof let us show that
\[
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon)), Du_\epsilon)) \, dx dt \to \int_0^T \int_\Omega (b(Du), Du) \, dx dt. \tag{5.21}
\]
First we observe that
\[
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), Du_\epsilon)) \, dx dt = - \langle u'_\epsilon, u_\epsilon \rangle + \langle f, u_\epsilon \rangle,
\]
or equivalently
\[
\int_0^T \int_\Omega (a_\epsilon(x,t,Du_\epsilon), Du_\epsilon)) \, dx dt = - \frac{1}{2} (\|u_\epsilon(T)\|^2_{L^2(\Omega)} - \|u_\epsilon(0)\|^2_{L^2(\Omega)}) + \langle f, u_\epsilon \rangle.
Since $W_0$ is continuously embedded in $C(0,T;L^2(\Omega))$ we can pass to the limit in the right hand side and, consequently,

$$\int_0^T \int_\Omega (b(Du), Du) \, dx \, dt = -\frac{1}{2} \left( \|u(T)\|_{L^2(\Omega)}^2 - \|u(0)\|_{L^2(\Omega)}^2 \right) + \langle f, u \rangle.$$

By collecting the results from the Steps 1-4 (5.21) follows and the proof is complete. ■

**Remark 4.** The results of Theorem 3.1 remain valid even for non-homogeneous or even more general boundary data. This follows from Theorem 6.1 in [9]. We can also allow oscillating right hand side and initial data, c.f. Theorem 4.1 and Remark 6.1 in [9].

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**References**


