Reflectionless Analytic Difference Operators II. Relations to Soliton Systems

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Abstract

This is the second part of a series of papers dealing with an extensive class of analytic difference operators admitting reflectionless eigenfunctions. In the first part, the pertinent difference operators and their reflectionless eigenfunctions are constructed from given "spectral data", in analogy with the IST for reflectionless Schrödinger and Jacobi operators. In the present paper, we introduce a suitable time dependence in the data, arriving at explicit solutions to a nonlocal evolution equation of Toda type, which may be viewed as an analog of the KdV and Toda lattice equations for the latter operators. As a corollary, we reobtain various known results concerning reflectionless Schrödinger and Jacobi operators. Exploiting a reparametrization in terms of relativistic Calogero–Moser systems, we also present a detailed study of N-soliton solutions to our nonlocal evolution equation.

1 Introduction

In our previous paper Ref. [1] we have introduced and studied an extensive class of AΔOs (analytic difference operators) admitting reflectionless eigenfunctions. Our construction in Ref. [1] is patterned after the IST for Schrödinger and Jacobi operators. In brief, we start from given spectral data (r, μ), and associate to these data a meromorphic reflectionless wave function $\mathcal{W}(x, p)$ and meromorphic coefficients $V_a(x)$, $V_b(x)$ ("potentials") of an AΔO A of the form

$$A \equiv T_i + V_a(x)T_{-i} + V_b(x), \qquad T_{\pm i} \equiv \exp(\mp i\partial_x), \tag{1.1}$$

such that we have the eigenvalue equation

$$(A\mathcal{W})(x, p) = \left(e^p + e^{-p}\right) \mathcal{W}(x, p). \tag{1.2}$$

The data $r = (r_1, \ldots, r_N)$ allowed in Ref. [1] (from now on denoted by Part I) gives rise to the poles of the transmission coefficient, just as for Schrödinger and Jacobi operators. It consists of complex numbers restricted by

$$\operatorname{Im} r_n \in (-\pi, 0) \cup (0, \pi), \qquad n = 1, \ldots, N, \tag{1.3}$$

and

$$e^{r_m} \neq e^{\pm r_n}, \qquad 1 \leq m < n \leq N. \tag{1.4}$$

This guarantees that the Cauchy matrix

$$C(r)_{mn} \equiv \frac{1}{e^{r_m} - e^{-r_n}}, \qquad m, n = 1, \ldots, N, \tag{1.5}$$

is well defined and regular, just as the Cauchy matrix pertinent to the Schrödinger and Jacobi cases. (Cf. Eqs. (2.30)–(2.33) in Part I, or, briefly, I(2.30)–(2.33).)

The "normalization coefficients" $\mu = (\mu_1, \ldots, \mu_N)$ permitted in Part I are however far more general than for reflectionless Schrödinger and Jacobi operators. Indeed, they are allowed to be meromorphic functions satisfying (cf. I(2.34))

$$\mu_n(x + i) = \mu(x), \qquad \lim_{|\mathrm{Re}\, x| \to \infty} \mu_n(x) = c_n, \qquad c_n \in \mathbb{C}^*, \qquad n = 1, \ldots, N. \tag{1.6}$$

Of course, this includes the constant case $\mu_n(x) = c_n$, which is the analog of the Schrödinger and Jacobi settings. In Part III of this series of papers [2] (which deals with various functional-analytic features), we focus attention on the constant multiplier case. In Sections 2 and 5 of the present paper, however, we allow the same multipliers as in Part I.

In Section 2 we introduce time dependence in the multipliers $\mu_1(x), \ldots, \mu_N(x)$. Accordingly, the potentials $V_a(x)$, $V_b(x)$ and wave function $\mathcal{W}(x, p)$ depend on time. The time dependence is chosen such that for each set of "initial" spectral data (r, μ), we arrive at a solution to a nonlocal Toda type equation. Specifically, this equation reads

$$\ddot{\Psi}(x, t) = i \exp(i[\Psi(x + i, t) - \Psi(x, t)]) - i \exp(i[\Psi(x, t) - \Psi(x - i, t)]). \tag{1.7}$$

Here, the function $\exp[i\Psi(x, t)]$, $x, t \in \mathbb{C}$, is assumed to be meromorphic in x for all t. With suitable restrictions on the data r, μ, we show that our solutions to (1.7) are real-valued, real-analytic functions for real x, t, with a solitonic long-time behavior (in Section 6).

Soliton equations of this nonlocal type have been introduced and studied before, cf. Santini's review Ref. [3] and references given there. (In particular, when one replaces the lhs of (1.7) by $i\partial_x \dot{\Psi}(x, t)$, then one obtains the so-called intermediate Toda lattice [4].) Even so, it seems that (1.7) is a new soliton equation, in the sense that there appears to be no obvious transformation relating it to previously known soliton equations.

We should stress at this point that we do not associate to (1.7) a clear-cut auxiliary linear spectral problem for self-adjoint AΔOs on $L^2(\mathbb{R}, dx)$. In fact, we view it as a challenging open problem to do so, in such a way that the self-adjoint AΔOs of Part III [2] arise as the ones with vanishing reflection.

In Sections 3 and 4 we present further results that render the existence of such a scenario plausible. Indeed, in these sections we show that our (constant multiplier) AΔOs can be related to the well-known reflectionless Schrödinger and Jacobi operators, respectively. Moreover, in Section 4 we show that the corresponding solutions to (1.7) obtained in Section 2 are related in the same way to the Toda lattice solitons.

Just as Part I [1], Sections 2–4 are basically self-contained, in the sense that we need not invoke any substantial previous results in the literature. (To be sure, the previous, "IST-based", literature in the KdV and Toda lattice settings did provide considerable inspiration

for this paper as well as Part I.) For example, we obtain the solution property for the Toda lattice solitons as a corollary of the corresponding result in Section 2.

In contrast, Section 5 (and Section 6, too) involves previous work of ours: We demonstrate there that the reflectionless wave functions and AΔOs can be connected to a special class of relativistic Calogero–Moser type systems. We have labeled the pertinent finite-dimensional soliton systems by $\tilde{\mathrm{II}}_{\mathrm{rel}}(\tau = \pi/2)$ in our paper [5]; more general integrable N-particle systems of Calogero–Moser and Toda type are surveyed in our lecture notes Ref. [6].

This relation plays an important role in Part III. It enables us to invoke various results from Ref. [5] to control analytical difficulties. Staying within the context of the present paper, it yields a map from an arbitrary point in the $\tilde{\mathrm{II}}_{\mathrm{rel}}(\tau = \pi/2)$ N-body phase space to a (real-valued) N-soliton solution of (1.7). As such, it gives rise to one more example of what we have dubbed soliton-particle correspondence in our survey Ref. [7]. In this connection we also mention previous results on this correspondence for the KdV and Toda lattice solitons [8], and more recent results on the relation between relativistic Calogero–Moser systems and the 2D Toda field theory [9, 10].

A novel feature of the correspondence in the present setting is that our general N-soliton solutions to (1.7) are encoded via the *defining* Lax matrix of the $\tilde{\mathrm{II}}_{\mathrm{rel}}(\tau = \pi/2)$ system, and not via the dual ("action-angle") Lax matrix (as is the case for the sine-Gordon and modified KdV particle-like solutions [5]). To be more specific, the general N-soliton solution consists of $N_+ \in \{0, 1, \ldots, N\}$ solitons moving to the right, and $N_- = N - N_+$ solitons moving to the left. The right-movers and left-movers are parametrized by the particle and antiparticle variables, respectively. When N_+ or N_- vanishes, one is dealing with a self-dual pair of Lax matrices, so this new feature plays no role.

For $N_+ N_- > 0$, however, self-duality breaks down. In that case the correspondence between right-/left-movers and particles/antiparticles is quite different from the correspondence between solitons, antisolitons and breathers in the sine-Gordon and modified KdV settings on the one hand, and particles, antiparticles, and their bound states in the $\tilde{\mathrm{II}}_{\mathrm{rel}}(\tau = \pi/2)$ system on the other hand.

The consequences of this novel type of correspondence are made explicit in Section 6, where we study the N-soliton solutions to (1.7). First of all, we demonstrate that the pertinent solutions deserve their name. Indeed, we show that for long times they can be approximated by linear combinations of N 1-soliton solutions; moreover, the asymptotic velocities are conserved and the position shifts are factorized in terms of pair shifts.

We obtain the features just mentioned in a quite direct and elementary fashion, yielding however a weaker approximation result than we obtained for the sine-Gordon and (m)KdV solitons in Section 7 of Ref. [5]. Just as in all cases studied previously, the long-time behavior of the N-soliton solutions is intimately related to the spectral asymptotics for $t \to \pm\infty$ of time-dependent matrices. When N_- or N_+ vanishes, we show that our results in *loc. cit.* give rise to a natural notion of global, non-intersecting, space-time trajectories for the N right-moving or N left-moving solitons.

For $N_+ N_- > 0$, however, we wind up with non-intersecting space-time trajectories only for sufficiently large times. In between, trajectories are ill-defined, since a collision of two trajectories typically gives rise to a complex-conjugate pair of eigenvalues. From a physical viewpoint, the right-moving soliton ("particle") and left-moving soliton ("antiparticle") involved in the collision form a resonance for a certain period of time.

2   A nonlocal Toda type soliton equation

We begin by collecting the definitions of various important quantities, cf. Section 2 in Part I. We have already recalled the restrictions on the spectral data (r, μ) and the definition of the Cauchy matrix $C(r)$, cf. (1.3)–(1.6). The dependence on μ is encoded in the diagonal matrix

$$D(r, \mu; x) \equiv \operatorname{diag}\left(d(r_1, \mu_1; x), \ldots, d(r_N, \mu_N; x)\right), \tag{2.1}$$

where the function d is defined by

$$d(\rho, \nu; x) \equiv \begin{cases} \nu(x)e^{-2i\rho x}, & \operatorname{Im} \rho \in (0, \pi), \\ \nu(x)e^{-2i(\rho + i\pi)x}, & \operatorname{Im} \rho \in (-\pi, 0). \end{cases} \tag{2.2}$$

To ease the notation, we often write

$$d_n(x) = d(r_n, \mu_n; x), \qquad n = 1, \ldots, N. \tag{2.3}$$

All of the remaining quantities can now be defined in terms of the solution $R(r, \mu; x)$ to the linear system

$$(D(r, \mu; x) + C(r))R(x) = \zeta, \qquad \zeta \equiv (1, \ldots, 1)^t. \tag{2.4}$$

Specifically, introducing the auxiliary functions

$$\lambda(r, \mu; x) \equiv 1 + \sum_{n=1}^{N} e^{r_n} R_n(r, \mu; x), \tag{2.5}$$

$$\Sigma(r, \mu; x) \equiv \sum_{n=1}^{N} R_n(r, \mu; x), \tag{2.6}$$

the potentials are given by

$$V_a(r, \mu; x) \equiv \lambda(r, \mu; x)/\lambda(r, \mu; x + i), \tag{2.7}$$

$$V_b(r, \mu; x) \equiv \Sigma(r, \mu; x - i) - \Sigma(r, \mu; x), \tag{2.8}$$

and the wave function reads

$$\mathcal{W}(r, \mu; x, p) \equiv e^{ixp}\left(1 - \sum_{n=1}^{N} \frac{R_n(r, \mu; x)}{e^p - e^{-r_n}}\right). \tag{2.9}$$

We now introduce time-dependent multipliers

$$\mu_n(r_n; x, t) \equiv \mu_n(x) \exp\left(it\left[e^{r_n} - e^{-r_n}\right]\right), \qquad n = 1, \ldots, N. \tag{2.10}$$

Correspondingly, the above quantities (2.1)–(2.9) are henceforth viewed as depending on time as well. But as a rule we suppress the dependence on t, just as the dependence on (r, μ). Until further notice, t may be viewed as a complex parameter. We denote partial differentiation with respect to t by a dot. (The factor i in (2.10) occurs with an eye on later reality restrictions.)

We proceed by obtaining the time derivatives of the above quantities. The pertinent formulas are far from immediate, so for clarity we assemble them in a series of propositions.

Proposition 2.1. *One has*

$$\dot{R}(x) = i \operatorname{diag}\left(e^{-r_1}, \ldots, e^{-r_N}\right)(R(x) - R(x-i)) - iV_b(x)R(x). \tag{2.11}$$

Proof. Combining (2.1), (2.2) with (2.10), we obtain

$$\dot{D}(x) = iD_-D(x), \tag{2.12}$$

where we have introduced

$$D_- \equiv \operatorname{diag}\left(e^{r_1} - e^{-r_1}, \ldots, e^{r_N} - e^{-r_N}\right). \tag{2.13}$$

Now from (2.4) we deduce

$$\dot{D}(x)R(x) + (D(x) + C)\dot{R}(x) = 0, \tag{2.14}$$

so that

$$(D(x) + C)\dot{R}(x) = -iD_-D(x)R(x). \tag{2.15}$$

Therefore, (2.11) will follow once we show

$$D_-D(x)R(x) = (D(x) + C)\operatorname{diag}\left(e^{-r_1}, \ldots, e^{-r_N}\right)(R(x-i) - R(x)) + V_b(x)\zeta. \tag{2.16}$$

(We used (2.4) to simplify the rhs.)

To prove (2.16), consider its nth component. Canceling terms $-\exp(-r_n)d_n(x)R_n(x)$ on the lhs and rhs, it reads

$$e^{r_n}d_n(x)R_n(x) = d_n(x)e^{-r_n}R_n(x-i)$$
$$+ \sum_{j=1}^{N} C_{nj}e^{-r_j}(R_j(x-i) - R_j(x)) + V_b(x). \tag{2.17}$$

In view of (2.2) and (2.8), this can be rewritten as

$$e^{r_n}(d_n(x)R_n(x) - d_n(x-i)R_n(x-i))$$
$$= \sum_{j=1}^{N} \left(C_{nj}\left(e^{r_n} - \left[e^{r_n} - e^{-r_j}\right]\right) + 1\right)(R_j(x-i) - R_j(x)). \tag{2.18}$$

Now from the definition (1.5) of the Cauchy matrix we see that this amounts to

$$d_n(x)R_n(x) - d_n(x-i)R_n(x-i) = \sum_{j=1}^{N} C_{nj}(R_j(x-i) - R_j(x)). \tag{2.19}$$

By virtue of (2.4), this is clearly true, so (2.16) follows.   ∎

Proposition 2.2. *Introducing the $A\Delta O$*

$$B \equiv -i(T_i + V_b(x)), \tag{2.20}$$

one has

$$\dot{\mathcal{W}}(x,p) = (B\mathcal{W})(x,p) + ie^p\mathcal{W}(x,p). \tag{2.21}$$

Proof. From (2.9) we obtain

$$\dot{\mathcal{W}}(x, p) = -e^{ixp} \sum_{n=1}^{N} \frac{\dot{R}_n(x)}{e^p - e^{-r_n}}. \tag{2.22}$$

Using (2.11), this can be rewritten as

$$i\dot{\mathcal{W}}(x, p) = -e^{ixp}\Sigma(x) + e^p e^{ixp} \sum_{n=1}^{N} \frac{R_n(x)}{e^p - e^{-r_n}}$$

$$+ e^{ixp}\Sigma(x - i) - e^p e^{ixp} \sum_{n=1}^{N} \frac{R_n(x - i)}{e^p - e^{-r_n}} - V_b(x)(e^{ixp} - \mathcal{W}(x, p)). \tag{2.23}$$

From the definition (2.8) of V_b we now see that we can cancel three terms on the rhs. Then we are left with

$$i\dot{\mathcal{W}}(x, p) = -e^p(\mathcal{W}(x, p) - e^{ixp})$$

$$+ e^p \left(e^{-p}\mathcal{W}(x - i, p) - e^{ixp} \right) + V_b(x)\mathcal{W}(x, p), \tag{2.24}$$

which amounts to (2.21). ∎

Since the wave function satisfies the AΔE (1.2), and iB equals $A - V_a(x)T_{-i}$ (cf. (1.1)), an alternative formula for the time derivative reads

$$\dot{\mathcal{W}}(x, p) = iV_a(x)\mathcal{W}(x + i, p) - ie^{-p}\mathcal{W}(x, p). \tag{2.25}$$

We now proceed with the time derivatives of the potentials.

Proposition 2.3. *One has*

$$\dot{\Sigma}(x) = i(1 - V_a(x)), \tag{2.26}$$

$$\dot{V}_b(x) = i(V_a(x) - V_a(x - i)). \tag{2.27}$$

Proof. In view of (2.8), the time derivative (2.27) is immediate from (2.26). To prove (2.26), we first use the relation

$$\mathcal{W}(x, r_n) = e^{ixr_n}d_n(x)R_n(x), \tag{2.28}$$

and the $\mathcal{W}$-AΔE (1.2) to deduce

$$e^{-r_n}R_n(x - i) + e^{r_n}V_a(x)R_n(x + i) + \left(V_b(x) - e^{r_n} - e^{-r_n} \right)R_n(x) = 0. \tag{2.29}$$

Therefore, (2.11) can be rewritten as

$$i\dot{R}_n(x) = e^{r_n}(R_n(x) - V_a(x)R_n(x + i)), \qquad n = 1, \ldots, N. \tag{2.30}$$

Taking now the sum of these N equations and using (2.5)–(2.7), we obtain

$$i\dot{\Sigma}(x) = \lambda(x) - 1 - \frac{\lambda(x)}{\lambda(x + i)}[\lambda(x + i) - 1] = V_a(x) - 1, \tag{2.31}$$

as asserted. ∎

In order to obtain the time derivative of $V_a(x)$, and also for later purposes, it is expedient to introduce one more quantity, namely, the τ-function

$$\tau(r,\mu;x,t) \equiv \left| \mathbf{1}_N + C(r)D(r,\mu;x,t)^{-1} \right|. \tag{2.32}$$

In view of the alternative representation I(C.30) for $\lambda(x)$, we readily obtain

$$\lambda(x) = \tau(x-i)/\tau(x), \tag{2.33}$$

$$V_a(x) = \tau(x+i)\tau(x-i)/\tau(x)^2. \tag{2.34}$$

We are now prepared for our last proposition.

Proposition 2.4. *One has*

$$\dot{\tau}(x)/\tau(x) = -i\Sigma(x), \tag{2.35}$$

$$\dot{V}_a(x) = iV_a(x)(V_b(x+i) - V_b(x)). \tag{2.36}$$

Proof. Clearly, (2.36) follows from (2.34), (2.8) and (2.35). To prove (2.35), we first use (2.32) and Leibniz' rule to write

$$\dot{\tau}(x) = -i\sum_{n=1}^{N} \tau_n(x). \tag{2.37}$$

Here, $\tau_n(x)$ denotes the determinant of the matrix obtained from $\mathbf{1}_N + CD(x)^{-1}$ when the nth column is replaced by $(e^{r_n} - e^{-r_n})\,d_n(x)^{-1}(C_{1n}, \ldots, C_{Nn})^t$. In the determinant quotient $\tau_n(x)/\tau(x)$ we now multiply both matrices from the right by $D(x)$. Then we obtain $|\Omega_n(x)|/|D(x) + C|$, with $\Omega_n(x)$ the matrix defined in the paragraph of Part I containing (C.6). From I(C.7) we then deduce (2.35). ∎

We now turn to some immediate consequences of the above formulas. First, we have from (1.1), (2.27) and (2.36)

$$\dot{A} = \dot{V}_a(x)T_{-i} + \dot{V}_b(x) = iV_a(x)(V_b(x+i) - V_b(x))T_{-i}$$
$$+ i(V_a(x) - V_a(x-i)) = i[V_a(x)T_{-i}, T_i + V_b(x)]. \tag{2.38}$$

Recalling (2.20), this can be rewritten as a Lax type equation

$$\dot{A} = [B, A]. \tag{2.39}$$

Second, from (2.35), (2.26) and (2.34) we deduce

$$\ddot{\tau}(x)\tau(x) - \dot{\tau}(x)^2 = \tau(x)^2 - \tau(x+i)\tau(x-i). \tag{2.40}$$

Third, introducing

$$\Psi(r,\mu;x,t) \equiv i\ln(\lambda(r,\mu;x,t)) = i\ln(\tau(r,\mu;x-i,t)/\tau(r,\mu;x,t)), \tag{2.41}$$

we readily obtain

$$\ddot{\Psi}(x) = i\exp(i[\Psi(x+i) - \Psi(x)]) - i\exp(i[\Psi(x) - \Psi(x-i)]). \tag{2.42}$$

The nonlocal evolution equation (2.42) is the Toda type equation announced in the Introduction, cf. (1.7). Since Lemma 2.2 in Part I yields

$$\lim_{\text{Re } x \to \infty} \lambda(x) = 1, \qquad \lim_{\text{Re } x \to -\infty} \lambda(x) = \exp\left(2\sum_{n=1}^{N} r_n\right), \tag{2.43}$$

it follows from (2.41) that we have

$$\lim_{\text{Re } x \to \infty} \Psi(x) = 0, \qquad \lim_{\text{Re } x \to -\infty} \Psi(x) = 2i\sum_{n=1}^{N} r_n, \qquad (\text{mod } 2\pi). \tag{2.44}$$

The multi-valuedness indicated here is inevitably present when we let x and t vary over $\mathbb{C}$. Indeed, $\tau(x,t)$ (2.32) has zeros in general (as well as poles whenever $\mu(x)$ is non-constant). Therefore, $\Psi(x,t)$ has logarithmic branch points. Note, however, that this multi-valuedness is of no consequence in (2.42).

We continue by studying reality restrictions. Specifically, we ask first: Can one choose the spectral data (r,μ) such that $\Psi(r,\mu;x,t)$ is real-valued for real x and t?

This question can be answered in the affirmative by using the results of Appendix D in Part I. We showed there that the AΔO A (1.1) is formally self-adjoint on $L^2(\mathbb{R}, dx)$ whenever $r_1, \ldots, r_N$ are purely imaginary and the functions $i\exp(-r_n)\mu_n(x)$, $n = 1, \ldots, N$, are real-valued for real x. Along the way, we obtained as another consequence of these restrictions the relation

$$\overline{\lambda(\overline{x})} = 1/\lambda(x), \qquad x \in \mathbb{C}, \tag{2.45}$$

cf. I(D.17). Imposing these restrictions and choosing t real from now on, the time-dependent factors $\exp(it[\exp(r_n) - \exp(-r_n)])$ belong to $(0,\infty)$, so we deduce

$$\overline{\lambda(x,t)} = 1/\lambda(x,t), \qquad x, t \in \mathbb{R}. \tag{2.46}$$

Now $\lambda(x,t)$ is meromorphic in x and t. For x, t real, (2.46) entails that $\lambda(x,t)$ is a phase factor, so in particular no zeros or poles occur. As a consequence, we need only fix the logarithm branch in (2.41) by requiring

$$\lim_{x \to \infty} \Psi(x,t) = 0, \qquad t \in \mathbb{R}, \tag{2.47}$$

to obtain a real-valued, real-analytic function

$$\Psi \,:\, \mathbb{R}^2 \to \mathbb{R}, \qquad (x,t) \mapsto \Psi(x,t). \tag{2.48}$$

Keeping t real, its (multi-valued) continuation to complex x satisfies

$$\overline{\Psi(t,\overline{x})} = \Psi(t,x), \qquad (\text{mod } 2\pi). \tag{2.49}$$

Let us now consider the characteristics of the solutions to (2.42) with the above reality restrictions in force, i.e.,

$$\text{Re } r_n = 0, \qquad n = 1, \ldots, N, \tag{2.50}$$

$$\text{Re }\left(e^{-r_n}\mu_n(x)\right) = 0, \qquad x \in \mathbb{R}, \qquad n = 1, \ldots, N. \tag{2.51}$$

Taking first $N = 1$, and choosing $r_1 = i\kappa^+$, $\kappa^+ \in (0, \pi)$, we get from (2.32) and (2.41)

$$\tau(x,t) = 1 + (2i \sin \kappa^+)^{-1} \exp(-2\kappa^+ x + 2t \sin \kappa^+)\mu_1(x)^{-1}, \tag{2.52}$$

$$\Psi(x,t) = i \ln \left(\frac{2ie^{-i\kappa^+}\mu_1(x) \sin \kappa^+ + e^{i\kappa^+} \exp(-2\kappa^+ x + 2t \sin \kappa^+)}{2ie^{-i\kappa^+}\mu_1(x) \sin \kappa^+ + e^{-i\kappa^+} \exp(-2\kappa^+ x + 2t \sin \kappa^+)} \right). \tag{2.53}$$

Since the function $ie^{-i\kappa^+}\mu_1(x)$ is real-valued for real x, this indeed yields a real-valued function for $(x,t) \in \mathbb{R}^2$. But whenever $\mu_1(x)$ is non-constant, $\Psi(x,t)$ cannot be viewed as a 1-soliton solution. Indeed, in that case it is not of the traveling wave form $f(x - vt)$.

Choosing however

$$2ie^{-i\kappa^+}\mu_1(x) \sin \kappa^+ = \exp(-2\kappa^+ a^+), \qquad a^+ \in \mathbb{R}, \tag{2.54}$$

we do get a function that is not only of the form $f(x - vt)$, but also of the kink type, in the sense that its x-derivative is positive and exponentially localized around its maximum at $x = a^+ + vt$:

$$\Psi'(x,t) = \frac{2\kappa^+ \sin \kappa^+}{\cos \kappa^+ + \cosh 2\kappa^+(x - a^+ - v(\kappa^+)t)}, \qquad v(\kappa) \equiv \frac{\sin \kappa}{\kappa}. \tag{2.55}$$

It is readily checked that the velocity function $v(\kappa)$ decreases monotonically from 1 to 0 as κ goes from 0 to π. Observe also that one has

$$\lim_{x \to -\infty} \Psi(x,t) = -2\kappa^+, \qquad t \in \mathbb{R}. \tag{2.56}$$

Consider next the choice $r_1 = i\kappa^- - i\pi$, $\kappa^- \in (0, \pi)$. Then we obtain

$$\tau(x,t) = 1 - (2i \sin \kappa^-)^{-1} \exp(-2\kappa^- x - 2t \sin \kappa^-)\mu_1(x)^{-1}, \tag{2.57}$$

$$\Psi(x,t) = i \ln \left(\frac{-2ie^{-i\kappa^-}\mu_1(x) \sin \kappa^- + e^{i\kappa^-} \exp(-2\kappa^- x - 2t \sin \kappa^-)}{-2ie^{-i\kappa^-}\mu_1(x) \sin \kappa^- + e^{-i\kappa^-} \exp(-2\kappa^- x - 2t \sin \kappa^-)} \right). \tag{2.58}$$

Once again, we need $\mu_1(x)$ to be constant for this function to have 1-soliton characteristics. Setting

$$-2ie^{-i\kappa^-}\mu_1(x) \sin \kappa^- = \exp(-2\kappa^- a^-), \qquad a^- \in \mathbb{R}, \tag{2.59}$$

we now obtain

$$\Psi'(x,t) = \frac{2\kappa^- \sin \kappa^-}{\cos \kappa^- + \cosh 2\kappa^-(x - a^- + v(\kappa^-)t)}. \tag{2.60}$$

Thus, $\Psi(x,t)$ is a 1-kink solution moving to the left with speed in the interval $(0,1)$. Clearly, it satisfies

$$\lim_{x \to -\infty} \Psi(x,t) = -2\kappa^-, \qquad t \in \mathbb{R}. \tag{2.61}$$

Proceeding with the arbitrary-N case, we can clearly make choices for $\mu_n(x)$ corresponding to (2.54) and (2.59), depending on whether r_n belongs to $i(0, \pi)$ or $i(-\pi, 0)$. Doing so, we obtain real-valued, real-analytic solutions. In Section 6 we study these solutions in detail, demonstrating in particular that they may be viewed as N-soliton solutions.

3 Reflectionless self-adjoint Schrödinger operators

In this section we clarify the relation of the above AΔOs and their reflectionless eigenfunctions to the reflectionless self-adjoint Schrödinger operators considered in the IST framework [11, 12, 13]. Thus, we start from a continuous real-valued potential $V_H(x)$ with decay at $\pm\infty$ given by

$$V_H(x) = O\left(|x|^{-e}\right), \qquad x \to \pm\infty, \tag{3.1}$$

with a suitable positive exponent e. Then it is clear that the Schrödinger operator on $L^2(\mathbb{R}, dx)$ given by

$$(Hf)(x) \equiv -f''(x) + V_H(x)f(x), \tag{3.2}$$

is self-adjoint on the natural domain of the free Hamiltonian $-d^2/dx^2$.

The reflectionless operators can now be characterized by the existence of an H-eigenfunction

$$(H\mathcal{W}_H)(x, p) = p^2 \mathcal{W}_H(x, p), \tag{3.3}$$

with asymptotics

$$\mathcal{W}_H(x, p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ a_H(p)\exp(ixp), & x \to -\infty. \end{cases} \tag{3.4}$$

The IST framework yields a complete classification of such operators: The function $a_H(p)$ is of the form

$$a_H(p) = \prod_{n=1}^{N} \frac{p - i\kappa_n}{p + i\kappa_n}, \qquad 0 < \kappa_N < \cdots < \kappa_1, \tag{3.5}$$

and for each such S-matrix there exists an N-dimensional family of potentials parametrized by normalization coefficients $\nu_1, \ldots, \nu_N \in (0, \infty)$. The eigenfunction $\mathcal{W}_H(x, p)$ yields bound states

$$\phi_n(\cdot) \equiv \mathcal{W}_H(\cdot, i\kappa_n), \qquad n = 1, \ldots, N, \tag{3.6}$$

satisfying

$$\int_{-\infty}^{\infty} |\phi_n(x)|^2 dx = 1/\nu_n, \qquad n = 1, \ldots, N. \tag{3.7}$$

Thus H has continuous spectrum $[0, \infty)$ and discrete spectrum $-\kappa_1^2, \ldots, -\kappa_N^2$. The bound states are pairwise orthogonal, and the improper eigenfunction $\mathcal{W}_H(x, p), p \in \mathbb{R}$, gives rise to the (Schwartz) kernel of an isometry from $L^2(\mathbb{R}, dp)$ onto the orthocomplement of the bound state subspace.

As we will show in Part III, the state of affairs concerning Hilbert space properties of the above AΔOs and their reflectionless eigenfunctions deviates from this "Schrödinger scenario" in several ways. In this section, however, we only aim to show how the self-adjoint reflectionless operators H and their eigenfunctions $\mathcal{W}_H(x, p)$ arise as limits of our AΔOs and their reflectionless eigenfunctions.

Turning to the details, we fix

$$(\kappa, \nu) \in (0, \infty)^{2N}, \qquad 0 < \kappa_N < \cdots < \kappa_1. \tag{3.8}$$

Then we choose

$$r_n = i\beta\kappa_n, \qquad \beta \in (0, \pi/\kappa_1), \qquad n = 1, \dots, N, \tag{3.9}$$

and constant multipliers

$$\mu_n(x) = -ie^{r_n}/\beta\nu_n, \qquad n = 1, \dots, N. \tag{3.10}$$

This entails that the AΔO A (1.1) is formally self-adjoint, cf. Theorem D.1 in Part I. Moreover, since all numbers $r_1, \dots, r_N$ have imaginary part in $(0, \pi)$, we may invoke Theorem 3.3 in Part I. It entails in particular that A can be rewritten as

$$A = S_+^2 - 2, \tag{3.11}$$

where

$$S_+ = T_{i/2} + V(x)T_{-i/2}, \tag{3.12}$$

$$V(x) = \sum_{n=1}^{N} (R_n(x - i/2) - R_n(x)) + 1. \tag{3.13}$$

Furthermore, the wave function is an S_+-eigenfunction:

$$(S_+ \mathcal{W})(x, p) = \left(e^{p/2} + e^{-p/2} \right) \mathcal{W}(x, p). \tag{3.14}$$

We now introduce a scaling $x \to \beta^{-1}x$, $p \to \beta p$ in the various quantities at hand, and study the $\beta \to 0$ limit. Specifically, introducing first

$$R_{\beta,n}(x) \equiv \beta^{-1} R_n \left(\beta^{-1}x \right), \tag{3.15}$$

the system (2.4) with $x \to \beta^{-1}x$ can be rewritten as

$$\frac{\exp(i\beta\kappa_n + 2\kappa_n x)}{i\nu_n} R_{\beta,n}(x)$$
$$+ \sum_{j=1}^{N} \frac{\beta}{\exp(i\beta\kappa_n) - \exp(-i\beta\kappa_j)} R_{\beta,j}(x) = 1, \qquad n = 1, \dots, N. \tag{3.16}$$

From this one readily deduces that $R_{\beta,n}(x)$ is holomorphic at $\beta = 0$, with a limit

$$\lim_{\beta \to 0} R_{\beta,n}(x) \equiv R_n^H(x), \qquad n = 1, \dots, N, \tag{3.17}$$

that solves the system

$$\frac{\exp(2\kappa_n x)}{i\nu_n} R_n^H(x) + \sum_{j=1}^{N} \frac{1}{i\kappa_n + i\kappa_j} R_j^H(x) = 1, \qquad n = 1, \dots, N. \tag{3.18}$$

Secondly, setting

$$\mathcal{W}_\beta(x,p) \equiv \mathcal{W}\left(\beta^{-1}x, \beta p\right),\tag{3.19}$$

it follows that $\mathcal{W}_\beta(x,p)$ is holomorphic at $\beta = 0$ as well, with a limit

$$\lim_{\beta \to 0} \mathcal{W}_\beta(x,p) = e^{ixp}\left(1 - \sum_{j=1}^{N} \frac{R_j^H(x)}{p + i\kappa_j}\right) \equiv \mathcal{W}_H(x,p).\tag{3.20}$$

Thirdly, introducing

$$S_{\beta,+} \equiv \exp(-i\beta\partial_x/2) + V_\beta(x)\exp(i\beta\partial_x/2),\tag{3.21}$$

with

$$V_\beta(x) \equiv \beta \sum_{n=1}^{N}(R_{\beta,n}(x - i\beta/2) - R_{\beta,n}(x)) + 1,\tag{3.22}$$

we clearly have

$$S_{\beta,+}\mathcal{W}_\beta(x,p) = \left(e^{\beta p/2} + e^{-\beta p/2}\right)\mathcal{W}_\beta(x,p).\tag{3.23}$$

Now from (3.17) we deduce

$$V_\beta(x) = 1 + (\beta/2)^2 V_H(x) + O\left(\beta^3\right), \qquad \beta \to 0,\tag{3.24}$$

where

$$V_H(x) \equiv -2i\sum_{n=1}^{N}\partial_x R_n^H(x).\tag{3.25}$$

As a consequence, we obtain

$$S_{\beta,+} = 2 + (\beta/2)^2 H + O\left(\beta^3\right), \qquad \beta \to 0,\tag{3.26}$$

with

$$H = -\partial_x^2 + V_H(x),\tag{3.27}$$

and

$$(H\mathcal{W}_H)(x,p) = p^2 \mathcal{W}_H(x,p).\tag{3.28}$$

Proceeding in this way, we have actually obtained all of the reflectionless self-adjoint Schrödinger operators delineated at the beginning of this section. To transform them to a more familiar form, we need only invoke (3.20) and the system (3.18) to write

$$\begin{aligned}\phi_n(x) &\equiv \mathcal{W}_H(x, i\kappa_n) = e^{-\kappa_n x}\left(1 - \sum_{j=1}^{N}\frac{R_n^H(x)}{i\kappa_n + i\kappa_j}\right)\\ &= \frac{e^{\kappa_n x}}{i\nu_n}R_n^H(x), \qquad n = 1,\dots,N.\end{aligned}\tag{3.29}$$

Thus $V_H(x)$ (3.25) can be rewritten as

$$V_H(x) = 2 \sum_{n=1}^{N} \nu_n \partial_x \left(e^{-\kappa_n x} \phi_n(x) \right). \tag{3.30}$$

This is the well-known formula expressing the reflectionless potentials in terms of their bound states $\phi_1(x), \ldots, \phi_N(x)$.

To conclude this section, we add three remarks. First, we point out that the asymptotics

$$\mathcal{W}(x,p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ a(p)\exp(ixp), & x \to -\infty, \end{cases} \tag{3.31}$$

$$a(p) = \prod_{n=1}^{N} \frac{e^p - e^{r_n}}{e^p - e^{-r_n}}, \tag{3.32}$$

derived in Theorem 2.3 of Part I yields the function $a_H(p)$ (3.5) when one substitutes $p \to \beta p$, $r_n \to i\beta\kappa_n$, and takes $\beta \to 0$. Second, we observe that we could also have used the AΔO

$$A_\beta \equiv S_{\beta,+}^2 - 2 \tag{3.33}$$

to arrive at the Schrödinger operator H, as will be clear from the above.

Finally, just as the operator H may be viewed as a reduced Hamiltonian for a Galilei-invariant two-particle system, one may view the AΔO $S_{\beta,+}$ (or A_β) as the reduced Hamiltonian for a Poincaré-invariant two-particle system. In this scenario the limit $\beta \to 0$ amounts to the nonrelativistic limit, cf. Ref. [14].

4   Reflectionless self-adjoint Jacobi operators and Toda lattice solitons

In this section we show how the reflectionless self-adjoint Jacobi operators and Toda lattice solitons arise via analytic continuation and x-discretization. For comparison purposes, especially useful are the monographs Refs. [15, 16] and Flaschka's paper Ref. [17]. For the Jacobi operator

$$(Jf)(n) = a(n-1)f(n-1) + a(n)f(n+1) + b(n)f(n) \tag{4.1}$$

on $l^2(\mathbb{Z})$ one requires decay

$$a(n) = 1/2 + O\left(|n|^{-e_a}\right), \qquad b(n) = O\left(|n|^{-e_b}\right), \qquad n \to \pm\infty, \tag{4.2}$$

for suitable positive exponents e_a, e_b. Requiring in addition that $a(n)$, $b(n)$ be real, it is clear that J is a bounded self-adjoint operator on $l^2(\mathbb{Z})$.

With these requirements in effect, the reflectionless Jacobi operators are characterized by the existence of a J-eigenfunction $\mathcal{W}_J(n,p)$ satisfying

$$(J\mathcal{W}_J)(n,p) = \cos(p)\mathcal{W}_J(n,p), \tag{4.3}$$

with asymptotics

$$\mathcal{W}_J(n,p) \sim \exp(inp), \qquad n \to \infty, \tag{4.4}$$

$$\mathcal{W}_J(n,p) \sim a_J(p)\exp(inp), \qquad n \to -\infty. \tag{4.5}$$

From the IST formalism it follows that the function $a_J(p)$ is of the form

$$a_J(p) = \prod_{j=1}^{N_+} \frac{\sin[(p - i\kappa_j^+)/2]}{\sin[(p + i\kappa_j^+)/2]} \cdot \prod_{l=1}^{N_-} \frac{\cos[(p - i\kappa_l^-)/2]}{\cos[(p + i\kappa_l^-)/2]}, \tag{4.6}$$

where $0 < \kappa_{N_\delta}^\delta < \cdots < \kappa_1^\delta$, $\delta = +, -$. For each $a_J(p)$ there exists an $(N_+ + N_-)$-dimensional family of Jacobi operators, parametrized by positive normalization coefficients $\nu_1^\delta, \ldots, \nu_{N_\delta}^\delta, \delta = +, -$. The operator J has $N_+ + N_-$ bound states, given by

$$\phi_j^+(n) = \mathcal{W}_J(n, i\kappa_j^+), \qquad j = 1, \ldots, N_+,$$
$$\phi_l^-(n) = \mathcal{W}_J(n, \pi + i\kappa_l^-), \qquad l = 1, \ldots, N_-, \tag{4.7}$$

and normalized as

$$\sum_{n \in \mathbb{Z}} \left| \phi_k^\delta(n) \right|^2 = 1/\nu_k^\delta, \qquad k = 1, \ldots, N_\delta, \qquad \delta = +, -. \tag{4.8}$$

The bound state energies equal $\delta \cosh(\kappa_j^\delta)$, $j = 1, \ldots, N_\delta$, $\delta = +, -$.

Fixing the above spectral data, we choose

$$N \equiv N_+ + N_-, \tag{4.9}$$

and let

$$r_j = e^{i\eta}\kappa_j^+, \qquad j = 1, \ldots, N_+, \qquad r_{N_+ + l} = e^{i\eta}\kappa_l^- - i\pi, \qquad l = 1, \ldots, N_-. \tag{4.10}$$

Here, we choose $\eta \in (0, \eta_s)$, with $\eta_s \in (0, \pi/2)$ satisfying $\kappa_1^\delta \sin \eta_s < \pi$, $\delta = +, -$. Then the requirements (1.3) and (1.4) are clearly met. We also choose constant multipliers

$$\mu_j(x) = \exp(r_j)/\nu_j^+, \qquad j = 1, \ldots, N_+,$$
$$\mu_{N_+ + l}(x) = \exp(r_{N_+ + l})/\nu_l^-, \qquad l = 1, \ldots, N_-. \tag{4.11}$$

Next, we substitute

$$x \to ie^{-i\eta}n, \qquad p \to -ie^{i\eta}p, \qquad n \in \mathbb{Z}, \qquad \eta \in (0, \eta_s), \tag{4.12}$$

in the above quantities, and study the $\eta \to 0$ limit. To this end, it is convenient (both for notation and for comparison purposes) to introduce further parameters

$$z_j \equiv \begin{cases} \exp(-\kappa_j^+), & \\ -\exp(-\kappa_{j-N_+}^-), & \end{cases} \qquad \nu_j \equiv \begin{cases} \nu_j^+, & j = 1, \ldots, N_+, \\ \nu_{j-N_+}^-, & j = N_+ + 1, \ldots, N. \end{cases} \tag{4.13}$$

Employing these parameters, one easily checks

$$\lim_{\eta \to 0} D\left(ie^{-i\eta}n\right) = \mathrm{diag}\left(1/z_1^{2n+1}\nu_1, \ldots, 1/z_N^{2n+1}\nu_N\right) \equiv D^J(n), \tag{4.14}$$

$$\lim_{\eta \to 0} C_{jk} = \frac{z_j}{1 - z_j z_k} \equiv C_{jk}^J / z_k, \qquad j, k = 1, \ldots, N. \tag{4.15}$$

Using $z_j \in (-1, 1)$, the matrix C^J is readily seen to be positive. (This can be deduced from Cauchy's identity. Alternatively, following Flaschka [17], one need only write $1/(1 - z_j z_k)$ as a geometric series to verify positivity.)

As a consequence, the matrix

$$L^J(n) \equiv \lim_{\eta \to 0} CD\left(ie^{-i\eta}n\right)^{-1}, \qquad n \in \mathbb{Z}, \tag{4.16}$$

is given by

$$L^J(n)_{jk} = C_{jk}^J z_k^{2n} \nu_k, \qquad j, k = 1, \ldots, N, \qquad n \in \mathbb{Z}. \tag{4.17}$$

Therefore, $L^J(n)$ is similar to a positive matrix, so $\mathbf{1}_N + L^J(n)$ is invertible. From this and the limit (4.14) it readily follows that

$$\lim_{\eta \to 0} R_j\left(ie^{-i\eta}n\right) \equiv R_j^J(n) \tag{4.18}$$

exists for all $n \in \mathbb{Z}$. Since $R_j^J(n)$ solves a system

$$\left(z_j^{2n+1}\nu_j\right)^{-1} R_j^J(n) + \sum_{k=1}^{N} \frac{z_j}{1 - z_j z_k} R_k^J(n) = 1, \qquad j = 1, \ldots, N, \tag{4.19}$$

with real coefficients, it is real, too.

Next, one easily verifies

$$\lim_{\eta \to 0} D\left(ie^{-i\eta}n \pm i\right) = D^J(n \pm i), \qquad n \in \mathbb{Z}. \tag{4.20}$$

From this it is not hard to deduce

$$\lim_{\eta \to 0} R_j\left(ie^{-i\eta}n \pm i\right) = R_j^J(n \pm i), \qquad j = 1, \ldots, N, \qquad n \in \mathbb{Z}. \tag{4.21}$$

(Indeed, the limit of the system (2.4) in small complex neighborhoods of $x = in \pm i$ can be controlled by exploiting (4.19) with $n \to n \pm 1$.)

Using the above, the remaining pertinent limits can be easily obtained. First, recalling (2.32)–(2.34) (with $t = 0$), one gets

$$\lim_{\eta \to 0} \tau\left(ie^{-i\eta}n\right) = \left|\mathbf{1}_N + L^J(n)\right| \equiv \tau^J(n), \tag{4.22}$$

$$\lim_{\eta \to 0} \lambda\left(ie^{-i\eta}n\right) = \tau^J(n-1)/\tau^J(n) \equiv \lambda^J(n), \tag{4.23}$$

$$\lim_{\eta \to 0} V_a\left(ie^{-i\eta}n\right) = \lambda^J(n)/\lambda^J(n+1) \equiv (2a(n))^2. \tag{4.24}$$

This defines $a(n) \in (0, \infty)$, since (4.22) entails $\tau^J(n) > 0$. Secondly, (2.8) yields

$$\lim_{\eta \to 0} V_b\left(ie^{-i\eta}n\right) = \sum_{j=1}^{N} \left(R_j^J(n-1) - R_j^J(n)\right) \equiv 2b(n). \tag{4.25}$$

Since $R_j^J(n)$ is real, $b(n)$ is real, too.

Thirdly, consider the wave function

$$\mathcal{W}_\eta(x, p) \equiv \mathcal{W}\left(ie^{-i\eta}x, -ie^{i\eta}p\right). \tag{4.26}$$

Its limit

$$\lim_{\eta \to 0} \mathcal{W}_\eta(n, p) = e^{inp}\left(1 - \sum_{j=1}^{N} \frac{R_j^J(n)}{e^{-ip} - z_j}\right) \equiv \hat{\mathcal{W}}^J(n, p), \qquad n \in \mathbb{Z}, \tag{4.27}$$

is once again immediate from the above. Now when we fix $n \in \mathbb{Z}$, the AΔE (1.2) yields

$$\mathcal{W}_\eta\left(n - e^{i\eta}, p\right) + V_a\left(ie^{-i\eta}n\right)\mathcal{W}_\eta\left(n + e^{i\eta}, p\right)$$
$$+ \left(V_b\left(ie^{-i\eta}n\right) - 2\cos\left(e^{i\eta}p\right)\right)\mathcal{W}_\eta(n, p) = 0. \tag{4.28}$$

Taking η to 0, we deduce that $\hat{\mathcal{W}}^J(n, p)$ satisfies the discrete difference equation

$$\hat{\mathcal{W}}^J(n - 1, p) + 4a(n)^2\hat{\mathcal{W}}^J(n + 1, p) + 2b(n)\hat{\mathcal{W}}^J(n, p) = 2\cos(p)\hat{\mathcal{W}}^J(n, p). \tag{4.29}$$

This is not yet of the Jacobi form (4.3), cf. (4.1), and we now proceed to explain how the connection is to be made.

Our reasoning involves the $|n| \to \infty$ asymptotics of the above quantities. This asymptotics does not follow from our previous work. For one thing, we have taken a limit that goes beyond the parameter regime studied in Part I. But even when one ignores this, the pertinent asymptotics is quite different from the one in Part I. Indeed, here we need the asymptotics in the direction of the shifts $n \to n \pm 1$, whereas in the AΔO context we are dealing with the asymptotics Re $x \to \pm\infty$, which is orthogonal to the shifts $x \to x \pm i$ in (1.1).

It is however a simple matter to obtain the desired $n \to \infty$ asymptotics directly. Indeed, from (4.19) and (4.13) one readily deduces

$$R_j^J(n) = O(\exp(-\rho n)), \qquad n \to \infty, \quad j = 1, \ldots, N, \quad \rho \equiv 2\min(\kappa_{N_+}^+, \kappa_{N_-}^-). \tag{4.30}$$

Likewise, from (4.17) one has

$$L^J(n)_{jk} = O(\exp(-\rho n)), \qquad n \to \infty, \qquad j, k = 1, \ldots, N. \tag{4.31}$$

Then (4.22) yields

$$\tau^J(n) = 1 + O(\exp(-\rho n)), \qquad n \to \infty, \tag{4.32}$$

so from (4.23) and (4.24) one obtains

$$\lambda^J(n) = 1 + O(\exp(-\rho n)), \qquad n \to \infty, \tag{4.33}$$

$$2a(n) = 1 + O(\exp(-\rho n)), \qquad n \to \infty. \tag{4.34}$$

Hence the infinite product

$$\Pi(n) \equiv \prod_{m=n}^{\infty} (2a(m))^{-1} \tag{4.35}$$

converges, and one has

$$\Pi(n) = \lambda^J(n)^{-1/2}. \tag{4.36}$$

When we now renormalize $\hat{\mathcal{W}}^J(n, p)$ by introducing

$$\mathcal{W}^J(n, p) \equiv \lambda^J(n)^{-1/2}\hat{\mathcal{W}}^J(n, p), \qquad n \in \mathbb{Z}, \tag{4.37}$$

then it follows from (4.29) that we have

$$a(n-1)\mathcal{W}^J(n-1, p) + a(n)\mathcal{W}^J(n+1, p) + b(n)\mathcal{W}^J(n, p) = \cos(p)\mathcal{W}^J(n, p), \tag{4.38}$$

which is of the Jacobi form (4.3). It is therefore clear from the above that the $n \to \infty$ asymptotics is in accord with (4.2) and (4.4): From (4.30) and (4.25) one gets

$$b(n) = O(\exp(-\rho n)), \qquad n \to \infty, \tag{4.39}$$

which, together with (4.34), agrees with (4.2) for $n \to \infty$. Moreover, (4.30), (4.27) and (4.34)–(4.37) entail (4.4).

It remains to show that the asymptotics for $n \to -\infty$ works out as announced. This involves a little more work. Let us first note that (4.19) entails that $R^J(n)$ has a finite limit $R^J(-\infty)$ for $n \to -\infty$, satisfying

$$\sum_{k=1}^{N} \frac{z_j}{1 - z_j z_k} R_k^J(-\infty) = 1, \qquad j = 1, \ldots, N. \tag{4.40}$$

Using Cramer's rule, this can be improved to

$$R_j^J(n) = R_j^J(-\infty) + O(\exp(\rho n)), \qquad n \to -\infty, \qquad j = 1, \ldots, N. \tag{4.41}$$

Therefore, (4.25) yields

$$b(n) = O(\exp(\rho n)), \qquad n \to -\infty. \tag{4.42}$$

Next, from (4.23), (4.22) and (4.17) we have

$$\lambda^J(n) = \frac{\left| \operatorname{diag}\left(z_1^{-2n}, \ldots, z_N^{-2n}\right) + \left(C_{jk}^J z_k^{-2} \nu_k\right) \right|}{\left| \operatorname{diag}\left(z_1^{-2n}, \ldots, z_N^{-2n}\right) + \left(C_{jk}^J \nu_k\right) \right|}$$

$$= \prod_{k=1}^{N} z_k^{-2} + O(\exp(\rho n)), \qquad n \to -\infty. \tag{4.43}$$

Therefore, we obtain from (4.24)

$$a(n) = 1/2 + O(\exp(\rho n)), \qquad n \to -\infty. \tag{4.44}$$

Summarizing, the asymptotics for $|n| \to \infty$ of the coefficients $a(n)$ and $b(n)$ agrees with (4.2).

We are left with determining the $n \to -\infty$ asymptotics of the wave function. A quick way to obtain this is to invoke the alternative representation for $\mathcal{W}(x, p)$ from Theorem C.3 of Part I. It entails that $\hat{\mathcal{W}}^J(n, p)$ (4.27) can also be written

$$\hat{\mathcal{W}}^J(n, p) = e^{inp} \frac{\left| D^J(n) + C^J \mathrm{diag}\left(z_1^{-1}, \ldots, z_N^{-1}\right) \Delta^J(p) \right|}{\left| D^J(n) + C^J \mathrm{diag}\left(z_1^{-1}, \ldots, z_N^{-1}\right) \right|}, \tag{4.45}$$

with

$$\Delta^J(p) \equiv \mathrm{diag}\left(\delta^J(z_1; p), \ldots, \delta^J(z_N; p)\right), \tag{4.46}$$

$$\delta^J(z; p) \equiv 1 - \frac{z^{-1} - z}{e^{-ip} - z} = \frac{e^{-ip} - z^{-1}}{e^{-ip} - z}. \tag{4.47}$$

Thus we have

$$\hat{\mathcal{W}}^J(n, p) \sim e^{inp} \prod_{j=1}^N \delta^J(z_j; p), \qquad n \to -\infty. \tag{4.48}$$

From (4.37) and (4.43) we now obtain

$$\mathcal{W}^J(n, p) \sim e^{inp} \prod_{j=1}^N |z_j| \frac{e^{-ip} - z_j^{-1}}{e^{-ip} - z_j}, \qquad n \to -\infty. \tag{4.49}$$

Substituting (4.13), this yields (4.5)–(4.6), as announced.

With the above $\eta \to 0$ limits at our disposal, it is straightforward to calculate the limits of the time-dependent quantities and time derivatives in Section 2. There is however one crucial change to be made before doing so: We should replace t by it so as to obtain the pertinent real-valued Toda lattice quantities for $\eta \to 0$.

Indeed, doing so in the formula (2.10) that defines the time-dependence, one obtains

$$\lim_{\eta \to 0} D(ie^{-i\eta}n, it) = \mathrm{diag}\left(1/z_1^{2n+1}\nu_1(t), \ldots, 1/z_N^{2n+1}\nu_N(t)\right) \equiv D^J(n, t), \tag{4.50}$$

$$\nu_j(t) \equiv \nu_j \exp\left(-t\left(z_j - z_j^{-1}\right)\right), \qquad j = 1, \ldots, N. \tag{4.51}$$

The time-dependent version of (4.16) then yields the limit

$$L^J(n, t)_{jk} = C_{jk}^J z_k^{2n} \nu_k(t), \qquad j, k = 1, \ldots, N, \qquad n \in \mathbb{Z}, \tag{4.52}$$

and (4.22) is generalized to

$$\lim_{\eta \to 0} \tau\left(ie^{-i\eta}n, it\right) = \left|\mathbf{1}_N + L^J(n, t)\right| \equiv \tau^J(n, t). \tag{4.53}$$

Then (2.40) entails Hirota's formula [18]

$$\partial_t^2 \ln\left(\tau^J(n, t)\right) = \frac{\tau^J(n+1, t)\tau^J(n-1, t)}{\tau^J(n, t)^2} - 1. \tag{4.54}$$

Likewise, (2.36) and (2.27), together with (4.24) and (4.25), yield

$$\dot{a}(n, t) = a(n, t)[b(n, t) - b(n+1, t)], \tag{4.55}$$

$$\dot{b}(n,t) = 2a(n-1,t)^2 - 2a(n,t)^2. \tag{4.56}$$

Comparing with Flaschka's paper Ref. [17], we see that (4.55), (4.56) coincide with his Eq. (2.3), up to signs. This sign difference arises from our different sign convention for the time-dependence in (4.51). (Our convention agrees with Refs. [15, 16].) Denoting the quantities he uses in his Section 3 with a superscript F, one gets (again up to irrelevant conventions)

$$\nu_j = \left(c_j^F\right)^2, \qquad j = 1, \ldots, N, \tag{4.57}$$

$$R_j^J(n) = -c_j^F z_j^{n+1} A_j^{Fn}, \qquad j = 1, \ldots, N, \qquad n \in \mathbb{Z}, \tag{4.58}$$

$$\lambda^J(n) = K^F(n,n)^{-2}. \tag{4.59}$$

(Compare his system Eq. (3.3) to (4.19) and note his Eq. (3.7) to check these correspondences.)

Various formulas in Refs. [15, 16] can also be obtained as $\eta \to 0$ limits. In this connection we mention in particular the Toda/Kac-van Moerbeke account in Section 3.8 of Toda's monograph Ref. [15]: The pertinent formulas can be readily derived by taking the $\eta \to 0$ limit of results that can be found in Section 3 of Part I.

Furthermore, we point out that van Diejen recently obtained the analog of the formula (4.45) for the Jacobi wave function $\mathcal{W}^J(n,p)$, cf. Ref. [19]. His paper also contains further results of interest, and information on recent literature dealing with Jacobi operators and the Toda/Kac-van Moerbeke correspondence.

5   Parametrization via relativistic Calogero–Moser systems

Returning to the general setting of Section 2, we proceed to detail the connection with the $\tilde{\Pi}_{\text{rel}}(\tau = \pi/2)$ systems from Ref. [5]. The connection to these finite-dimensional soliton systems hinges on a suitable reparametrization of the Cauchy matrix (1.5) and the multipliers in the diagonal matrix $D(r, \mu; x)$ (2.1). To ease the notation we assume from now on that the numbers $r_1, \ldots, r_N$ are ordered such that $r_1, \ldots, r_{N_+}$ have imaginary parts in $(0, \pi)$ and $r_{N-N_-+1}, \ldots, r_N$ in $(-\pi, 0)$, with

$$N_+ \in \{0, 1, \ldots, N\}, \qquad N_- = N - N_+. \tag{5.1}$$

(Since the wave function $\mathcal{W}(r, \mu; x, p)$ and AΔO $A(r, \mu)$ are invariant under permutations on the data (r, μ), this does not give rise to a loss of generality.)

It now turns out that the numbers $r_1, \ldots, r_{N_+}$ and $r_{N-N_-+1}, \ldots, r_N$ can be traded for the (complex) positions $q_1^+, \ldots, q_{N_+}^+$ and $q_1^-, \ldots, q_{N_-}^-$ of the particles and antiparticles, resp., in the $\tilde{\Pi}_{\text{rel}}$ system, in such a way that the reparametrized Cauchy matrix $C(r)$ (1.5) and the Cauchy matrix $\mathcal{C}$ in the $\tilde{\Pi}_{\text{rel}}$ Lax matrix are closely related. This relation then suggests a reparametrization of the multipliers in $D(x)$ such that the matrix $CD(x)^{-1}$ may be reinterpreted as the $\tilde{\Pi}_{\text{rel}}(\tau = \pi/2)$ Lax matrix evaluated in x-dependent points of the (complexified) $\tilde{\Pi}_{\text{rel}}(\tau = \pi/2)$ phase space.

More precisely, the latter identification holds true up to diagonal similarity transformations. Since we are dealing with determinants and spectra, such "gauge transformations"

are immaterial. In particular, the definition Eq. (2.70) of the Lax matrix in Ref. [5] amounts to a gauge choice that facilitates its spectral analysis, but in the present context another gauge is more convenient.

Specifically, here we work with the Lax matrix

$$\mathcal{L}(q, \theta) \equiv \mathcal{C}(q^+, q^-)\mathcal{D}(q^+, q^-, \theta^+, \theta^-). \tag{5.2}$$

The Cauchy matrix $\mathcal{C}$ is defined by

$$\mathcal{C}_{jk} \equiv 1/\cosh[(q_j^+ - q_k^+)/2], \tag{5.3}$$

$$\mathcal{C}_{N_++l,N_++m} \equiv 1/\cosh[(q_l^- - q_m^-)/2], \tag{5.4}$$

$$\mathcal{C}_{N_++l,k} \equiv -i/\sinh[(q_l^- - q_k^+)/2], \tag{5.5}$$

$$\mathcal{C}_{j,N_++m} \equiv i/\sinh[(q_j^+ - q_m^-)/2]. \tag{5.6}$$

Here and from now on, the indices j, k take values $1, \ldots, N_+$, whereas the indices l, m take values $1, \ldots, N_-$. The diagonal matrix $\mathcal{D}$ is defined by

$$\mathcal{D} \equiv \mathrm{diag}\left(\exp(\theta_1^+)V_1^+, \ldots, \exp(\theta_{N_+}^+)V_{N_+}^+, \exp(\theta_1^-)V_1^-, \ldots, \exp(\theta_{N_-}^-)V_{N_-}^-\right). \tag{5.7}$$

The quantities θ_n^δ are the generalized momenta corresponding to the positions q_n^δ, and the "potentials" V_n^δ are given by

$$V_j^+ \equiv \prod_{1\le k\le N_+, k\neq j}\left|\coth[(q_j^+ - q_k^+)/2]\right| \prod_{1\le l\le N_-}\left|\tanh[(q_j^+ - q_l^-)/2]\right|, \tag{5.8}$$

$$V_l^- \equiv \prod_{1\le m\le N_-, m\neq l}\left|\coth[(q_l^- - q_m^-)/2]\right| \prod_{1\le j\le N_+}\left|\tanh[(q_l^- - q_j^+)/2]\right|. \tag{5.9}$$

(The moduli we are choosing here preclude analyticity in q, but they enable us to steer clear of multi-valuedness issues. Such issues are important in other contexts, but here they would give rise to unnecessary complications.)

Comparing (5.2)–(5.9) with real q's and θ's to the Lax matrix L given by Eq. (2.70) in Ref. [5], we see that $\mathcal{L}$ amounts to a diagonal similarity transform, as announced. More precisely, we should substitute $\tau = \beta\mu g/2 = \pi/2$, $\mu = 1$, $\beta = 1$, and $x_n^\delta \to q_n^\delta$, $p_n^\delta \to \theta_n^\delta$ in *loc. cit.*, and choose distinct $q_1^+, \ldots, q_{N_-}^-$, cf. also Eqs. (1.3) and (1.4) in Ref. [5]. (To avoid confusion, we should add that the Cauchy matrix $\mathcal{C}$ (5.3)–(5.6) is slightly different from the matrix we refer to as Cauchy matrix in Ref. [5]: The latter equals $\mathcal{ECE}$, with

$$\mathcal{E} \equiv \mathrm{diag}\left(\exp(-q_1^+/2), \ldots, \exp(-q_{N_+}^+/2), \exp(-q_1^-/2), \ldots, \exp(-q_{N_-}^-/2)\right), \tag{5.10}$$

cf. also Eq. (B1) in Ref. [5].)

After these preliminaries, we turn to the reparametrizations of $C(r)$ and the multipliers in $D(r, \mu; x)$. To this end we first introduce parameters

$$\alpha_j^+ \equiv -ir_j, \tag{5.11}$$

$$\alpha_l^- \equiv -i r_{N_+ + l} + \pi, \tag{5.12}$$

which are convenient in their own right. Clearly, one has

$$\operatorname{Re} \alpha_n^\delta \in (0, \pi), \qquad n = 1, \ldots, N_\delta, \qquad \delta = +, -, \tag{5.13}$$

and the Cauchy matrix (1.5) can be rewritten as

$$C_{jk} = -\frac{i}{2} \exp(-i\alpha_j^+/2 + i\alpha_k^+/2) \frac{1}{\sin[(\alpha_j^+ + \alpha_k^+)/2]}, \tag{5.14}$$

$$C_{N_+ + l, N_+ + m} = \frac{i}{2} \exp(-i\alpha_l^-/2 + i\alpha_m^-/2) \frac{1}{\sin[(\alpha_l^- + \alpha_m^-)/2]}, \tag{5.15}$$

$$C_{N_+ + l, k} = -\frac{1}{2} \exp(-i\alpha_l^-/2 + i\alpha_k^+/2) \frac{1}{\cos[(\alpha_l^- + \alpha_k^+)/2]}, \tag{5.16}$$

$$C_{j, N_+ + m} = \frac{1}{2} \exp(-i\alpha_j^+/2 + i\alpha_m^-/2) \frac{1}{\cos[(\alpha_j^+ + \alpha_m^-)/2]}. \tag{5.17}$$

We also note that the restrictions (1.4) amount to

$$\alpha_j^+ \neq \alpha_k^+, \qquad j \neq k, \qquad \alpha_l^- \neq \alpha_m^-, \qquad l \neq m, \qquad \alpha_j^+ + \alpha_l^- \neq \pi. \tag{5.18}$$

(Recall $j, k \in \{1, \ldots, N_+\}$ and $l, m \in \{1, \ldots, N_-\}$.)

Next, we relate the quantities α_j^+ and α_l^- to the above positions q_j^+ and q_l^-, resp. To this end, we begin by pointing out that the map $\alpha \mapsto z = \cot(\alpha/2)$ yields an injection of the strip $\operatorname{Re} \alpha \in (0, \pi)$ onto the half plane $\operatorname{Re} z > 0$. (Even though this assertion is not immediate, it is straightforward to check.) Therefore, the map

$$F : \{\operatorname{Re} \alpha \in (0, \pi)\} \to \{\operatorname{Im} q \in (-\pi/2, \pi/2)\}, \qquad \alpha \mapsto q = \ln(\cot(\alpha/2)), \tag{5.19}$$

with $F(\pi/2) \equiv 0$, is holomorphic and has a holomorphic inverse. It is easily seen that the resulting relation

$$e^{-q} = \tan(\alpha/2), \qquad q = 0 \Leftrightarrow \alpha = \pi/2, \tag{5.20}$$

implies

$$\cosh q = 1/\sin \alpha, \qquad \sinh q = \cot \alpha, \qquad \tanh q = \cos \alpha. \tag{5.21}$$

We now set

$$q_n^\delta \equiv \delta F(\alpha_n^\delta), \qquad n = 1, \ldots, N_\delta, \qquad \delta = +, -. \tag{5.22}$$

Thus we have

$$\operatorname{Im} q_n^\delta \in (-\pi/2, \pi/2), \qquad n = 1, \ldots, N_\delta, \qquad \delta = +, -. \tag{5.23}$$

Moreover, the restrictions (5.18) translate into

$$q_j^+ \neq q_k^+, \quad j \neq k, \qquad q_l^- \neq q_m^-, \quad l \neq m, \qquad q_j^+ \neq q_l^-. \tag{5.24}$$

In words, the positions $q_1^+, \ldots, q_{N_+}^+, q_1^-, \ldots, q_{N_-}^-$ must be *distinct*.

When we combine the changes of variables $r \to (\alpha^+, \alpha^-) \to (q^+, q^-)$, we get from the above

$$\exp(r_j) = (i + \sinh q_j^+)/\cosh q_j^+, \tag{5.25}$$

$$\exp(r_{N_++l}) = (-i + \sinh q_l^-)/\cosh q_l^-. \tag{5.26}$$

But when we would substitute this directly in the Cauchy matrix (1.5), we would obtain a matrix whose relation to the Cauchy matrix $\mathcal{C}$ (5.3)–(5.6) is invisible. Instead, we start from (5.14) and use the relations (5.20) and (5.21) to calculate

$$\sin(\alpha_j^+ + \alpha_k^+)/2 = \cos(\alpha_j^+/2)\cos(\alpha_k^+/2)\left[\tan(\alpha_j^+/2) + \tan(\alpha_k^+/2)\right]$$

$$= \frac{1}{2}\left(\frac{\sin(\alpha_j^+)\sin(\alpha_k^+)}{\tan(\alpha_j^+/2)\tan(\alpha_k^+/2)}\right)^{1/2}\left[\tan(\alpha_j^+/2) + \tan(\alpha_k^+/2)\right] \tag{5.27}$$

$$= \frac{\cosh[(q_j^+ - q_k^+)/2]}{[\cosh(q_j^+)\cosh(q_k^+)]^{1/2}}.$$

(Note that the radicands have no zeros or poles in the pertinent regions, cf. (5.19). The branch choice is then clear: We need the positive square root for q^+ real.)

More generally, in terms of q^+, q^- the Cauchy matrix C (5.14)–(5.17) becomes

$$C = S\mathcal{C}S^{-1}D_c, \tag{5.28}$$

where

$$D_c \equiv \frac{i}{2}\operatorname{diag}\left(-\cosh q_1^+, \ldots, -\cosh q_{N_+}^+, \cosh q_1^-, \ldots, \cosh q_{N_-}^-\right), \tag{5.29}$$

$$S \equiv \operatorname{diag}\left(\exp(-i\alpha_1^+/2)[\cosh q_1^+]^{1/2}, \ldots, -\exp(-i\alpha_{N_-}^-/2)[\cosh q_{N_-}^-]^{1/2}\right). \tag{5.30}$$

We stick to the parameters α^+, α^- in the exponents, since this will be convenient shortly.

We now turn to the reparametrization of the multipliers in the diagonal matrix $D(x)$. In terms of α^+, α^-, this matrix reads

$$D(x) = \operatorname{diag}\left(\mu_1(x)\exp(2\alpha_1^+ x), \ldots, \mu_N(x)\exp(2\alpha_{N_-}^- x)\right), \tag{5.31}$$

cf. (2.1)–(2.2) and (5.11)–(5.12). Consider now the matrix

$$L(x + i/2) \equiv S^{-1}CD(x)^{-1}S = \mathcal{C}D_cD(x)^{-1}, \tag{5.32}$$

where we used (5.28). Comparing it to the Lax matrix $\mathcal{L}$ (5.2), we see that when we rewrite the multipliers as

$$\mu_j(x) = -\frac{i}{2}\exp(i\alpha_j^+)\cosh(q_j^+)\left[V_j^+\pi_j^+(x + i/2)\right]^{-1}, \tag{5.33}$$

$$\mu_{N_++l}(x) = \frac{i}{2}\exp(i\alpha_l^-)\cosh(q_l^-)\left[V_l^-\pi_l^-(x + i/2)\right]^{-1}, \tag{5.34}$$

then we have

$$L(x) = \mathcal{L}(q, \theta)\mathcal{M}(x), \tag{5.35}$$

with

$$\mathcal{M}(x) \equiv \mathrm{diag}\left(\exp(-\theta_1^+ - 2\alpha_1^+ x)\pi_1^+(x), \ldots, \exp(-\theta_{N_-}^- - 2\alpha_{N_-}^- x)\pi_{N_-}^-(x)\right). \tag{5.36}$$

The above formulas (5.28)–(5.36) encode the announced relation between the two key matrices C and $D(x)$ and the $\tilde{\mathrm{II}}_{\mathrm{rel}}(\tau = \pi/2)$ Lax matrix from Ref. [5]. Note that the parametrizations (5.33) and (5.34) give rise to well-defined i-periodic meromorphic multipliers $\pi_n^\delta(x)$, with finite limits

$$\lim_{|\mathrm{Re}\, x| \to \infty} \pi_n^\delta(x) \equiv \pi_n^\delta, \qquad n = 1, \ldots, N_\delta, \qquad \delta = +, -. \tag{5.37}$$

Admittedly, at this stage it is not clear that the relation just established is useful. At first sight, it merely seems a bizarre coincidence, and indeed the use of the position variables q_n^δ would have been quite inconvenient in Part I and in Section 2 of the present paper.

As it turns out, however, the relation can be used to great advantage, not only for studying N-soliton solutions (which we do in Section 6), but also for studying the above AΔOs from the viewpoint of quantum mechanics (which we do in Part III). For both of these applications, we restrict attention to purely imaginary $r_1, \ldots, r_N$, and to constant and positive multipliers π_n^δ. The associated positions q_n^δ are then real, and we may and will choose real θ_n^δ such that

$$\pi_n^\delta = \exp\left(\theta_n^\delta\right), \qquad n = 1, \ldots, N_\delta, \qquad \delta = +, -. \tag{5.38}$$

(Notice that these restrictions amount to the ones at the end of Section 2.)

With the choices just detailed in effect from now on, we obtain from each point in the phase space

$$\Omega \equiv \left\{ (q_1^+, \ldots, q_{N_-}^-, \theta_1^+, \ldots, \theta_{N_-}^-) \in \mathbb{R}^{2N} \mid q_{N_+}^+ < \cdots < q_1^+, \right.$$
$$\left. q_{N_-}^- < \cdots < q_1^-, q_j^+ \neq q_l^-, \ j = 1, \ldots, N_+, \ l = 1, \ldots, N_- \right\} \tag{5.39}$$

of the $\tilde{\mathrm{II}}_{\mathrm{rel}}(\tau = \pi/2)$ system [5] an AΔO A that is formally self-adjoint, and a real-valued, real-analytic solution to (1.7). We now turn to a study of the latter solutions.

6   A close-up of the N-soliton solutions

The τ-function (2.32) associated to a point (q, θ) in the phase space Ω (5.39) can be rewritten as

$$\tau(x, t) = |\mathbf{1}_N + L(x + i/2, t)| = |\mathbf{1}_N + L(x, t)U|, \tag{6.1}$$

with

$$L(x, t) \equiv \mathcal{L}\left(q^+, q^-, \theta_1^+ - 2\alpha_1^+[x - v(\alpha_1^+)t], \ldots, \theta_{N_-}^- - 2\alpha_{N_-}^-[x + v(\alpha_{N_-}^-)t]\right),$$
$$v(\alpha) \equiv \alpha^{-1}\sin\alpha, \tag{6.2}$$

$$U \equiv \mathrm{diag}\left(\exp(-i\alpha_1^+), \ldots, \exp(-i\alpha_{N_-}^-)\right). \tag{6.3}$$

(Recall (5.33)–(5.38) and (2.10) to see this.) The corresponding solution (2.41) to (2.42) then reads

$$\Psi(x,t) = i \ln\left(\frac{|\mathbf{1}_N + L(x,t)U^{-1}|}{|\mathbf{1}_N + L(x,t)U|}\right). \tag{6.4}$$

In particular, for $N_+ = 1, N_- = 0$, one obtains from (5.2)–(5.4) the right-moving soliton $f^+(q^+, \theta^+; x - v(\alpha^+)t)$, where

$$f^+(q,\theta;x) \equiv i \ln\left(\frac{1 + e^{i\alpha^+}\exp(\theta - 2\alpha^+ x)}{1 + e^{-i\alpha^+}\exp(\theta - 2\alpha^+ x)}\right), \tag{6.5}$$

$$\alpha^+(q) \equiv 2 \,\mathrm{Arctan}\,(\exp(-q)), \tag{6.6}$$

and for $N_+ = 0, N_- = 1$, the left-moving soliton $f^-(q^-, \theta^-; x + v(\alpha^-)t)$, where

$$f^-(q,\theta;x) \equiv i \ln\left(\frac{1 + e^{i\alpha^-}\exp(\theta - 2\alpha^- x)}{1 + e^{-i\alpha^-}\exp(\theta - 2\alpha^- x)}\right), \tag{6.7}$$

$$\alpha^-(q) \equiv 2 \,\mathrm{Arctan}\,(\exp(q)). \tag{6.8}$$

(We recall that we have already studied these solutions at the end of Section 2.) We proceed by showing that the general N-soliton solution $\Psi(x,t)$ (6.4) has a long-time asymptotics that is a linear superposition of N 1-soliton solutions.

To this end we define

$$\begin{aligned}
\Psi^{(\delta)}(x,t) \equiv \sum_{j=1}^{N_+} f^+\left(q_j^+, \theta_j^+; x + \delta\Delta_j^+(q^+, q^-)/4\alpha_j^+ - v(\alpha_j^+)t\right) \\
+ \sum_{l=1}^{N_-} f^-\left(q_l^-, \theta_l^-; x + \delta\Delta_l^-(q^+, q^-)/4\alpha_l^- + v(\alpha_l^-)t\right),
\end{aligned} \tag{6.9}$$

where $\delta = +, -$ and

$$\begin{aligned}
\Delta_j^+(q^+, q^-) \equiv \left(\sum_{k<j} - \sum_{k>j}\right) \ln\left(\coth^2[(q_j^+ - q_k^+)/2]\right), t) \\
- \sum_{l=1}^{N_-} \ln\left(\tanh^2[(q_j^+ - q_l^-)/2]\right),
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
\Delta_l^-(q^+, q^-) \equiv \left(\sum_{m<l} - \sum_{m>l}\right) \ln\left(\coth^2[(q_l^- - q_m^-)/2]\right) \\
+ \sum_{j=1}^{N_+} \ln\left(\tanh^2[(q_l^- - q_j^+)/2]\right).
\end{aligned} \tag{6.11}$$

Moreover, we introduce the remainder functions

$$R^{(\delta)}(x,t) \equiv \Psi(x,t) - \Psi^{(\delta)}(x,t), \qquad \delta = +, -. \tag{6.12}$$

Setting

$$\rho^{(\delta)}(t) \equiv \sup_{x \in \mathbb{R}} \left| R^{(\delta)}(x,t) \right|, \tag{6.13}$$

we conjecture that one has

$$\rho^{(\delta)}(t) = O(\exp(-\delta tr)), \qquad \delta t \to \infty, \qquad \delta = +, -, \qquad (?) \tag{6.14}$$

with

$$r \equiv \min_{j \neq k, l \neq m} \left(2\alpha_j^+ |v(\alpha_j^+) - v(\alpha_k^+)|, 2\alpha_j^+ (v(\alpha_j^+) + v(\alpha_l^-)), \right.$$
$$\left. 2\alpha_l^- |v(\alpha_l^-) - v(\alpha_m^-)|, 2\alpha_l^- (v(\alpha_l^-) + v(\alpha_j^+)) \right). \tag{6.15}$$

In Section 7 of Ref. [5] we proved the analog of this conjecture for the particle-like solutions to the sine-Gordon, modified KdV and KdV equations. Unfortunately, the strategy we followed in the latter cases does not apply here.

On the other hand, the structure of the solution $\Psi(x,t)$ (6.4) makes it possible to supply a quite simple and direct proof of a weaker convergence result, expressed in the following proposition.

Proposition 6.1. *Fixing $x_0, s_0 \in \mathbb{R}$, one has*

$$\lim_{\delta t \to \infty} \exp\left(-iR^{(\delta)}(x_0 + s_0 t, t)\right) = 1, \qquad \delta = +, -. \tag{6.16}$$

In order to appreciate this proposition and to prepare for its proof, we begin by pointing out that the function

$$F_t(x_0, s_0) \equiv f^+ \left(q, \theta; x_0 + s_0 t - v(\alpha^+) t\right) \tag{6.17}$$

has the following *discontinuous* limiting behavior:

$$\lim_{t \to \infty} F_t(x_0, s_0) = \begin{cases} 0, & s_0 > v(\alpha^+), \\ -2\alpha^+, & s_0 < v(\alpha^+), \\ f^+(q, \theta; x_0), & s_0 = v(\alpha^+). \end{cases} \tag{6.18}$$

(Indeed, this is clear from (6.5).) From this the $t \to \pm\infty$ limits of $\Psi^{(\pm)}(x_0 + s_0 t, t)$ (cf. (6.9)) are quite easily calculated. Denoting the limits by $F^{(\pm)}(x_0, s_0)$, Proposition 6.1 is equivalent to

$$\lim_{\delta t \to \infty} \Psi(x_0 + s_0 t, t) = F^{(\delta)}(x_0, s_0) + 2\pi k, \qquad \delta = +, -, \qquad k \in \mathbb{Z}. \tag{6.19}$$

A priori, the integer k depends on δ, x_0 and s_0, though it is undoubtedly true that one has $k = 0$ (recall the branch choice (2.47)). But we cannot rigorously deduce this, since (6.16) is a pointwise limit.

Next, we observe that the t-dependence of $\Psi(x_0+s_0t,t)$ is carried by factors $\exp(t\lambda_i(s_0))$ in the Lax matrix, with

$$\lambda_i(s) \equiv \begin{cases} -2\alpha_i^+(s-v(\alpha_i^+)), & i=1,\ldots,N_+, \\ -2\alpha_{i-N_+}^-(s+v(\alpha_{i-N_+}^-)), & i=N_++1,\ldots,N, \end{cases} \tag{6.20}$$

cf. (6.2). Furthermore, all principal minors of the matrices $LU^{\pm 1}$ occurring in (6.4) can be readily calculated via Cauchy's identity, yielding non-zero numbers.

We now turn to a lemma in which the state of affairs just sketched is studied in a somewhat more general setting. For M an $N \times N$ matrix, denote by M_j the $j \times j$ matrix obtained from M by deleting the rows and columns $j+1,\ldots,N$. Now we define a set $\mathcal{M}$ of matrices by

$$\mathcal{M} \equiv \{M \in M_N(\mathbb{C}) \mid |M_j| \neq 0, \ j=1,\ldots,N\}. \tag{6.21}$$

Lemma 6.2. *Let $M^+, M^- \in \mathcal{M}$ and let*

$$Q(t) \equiv \left|\mathbf{1}_N + M^+ e^{tD}\right| / \left|\mathbf{1}_N + M^- e^{tD}\right|, \tag{6.22}$$

where

$$D \equiv \operatorname{diag}(d_1,\ldots,d_N), \qquad d_1,\ldots,d_N \in \mathbb{R}. \tag{6.23}$$

Assuming

$$d_1,\ldots,d_n \in (0,\infty), \qquad d_{n+1},\ldots,d_N \in (-\infty,0), \qquad n \in \{0,\ldots,N\}, \tag{6.24}$$

one has

$$\lim_{t\to\infty} Q(t) = |M_n^+|/|M_n^-|, \tag{6.25}$$

with $|M_0| \equiv 1$. Assuming next

$$\begin{aligned} d_1,\ldots,d_n \in (0,\infty), & \qquad d_{n+1}=0, \\ d_{n+2},\ldots,d_N \in (-\infty,0), & \qquad n \in \{0,\ldots,N-1\}, \end{aligned} \tag{6.26}$$

one has

$$\lim_{t\to\infty} Q(t) = (|M_n^+|+|M_{n+1}^+|)/(|M_n^-|+|M_{n+1}^-|). \tag{6.27}$$

Proof of Lemma 6.2. We may write

$$Q(t) = \eta^+(t)/\eta^-(t), \tag{6.28}$$

where

$$\begin{aligned} \eta^\delta(t) \equiv \Big| &\operatorname{diag}\left(e^{-td_1},\ldots,e^{-td_n},1,\ldots,1\right) \\ &+ M^\delta \operatorname{diag}\left(1,\ldots,1,e^{td_{n+1}},\ldots,e^{td_N}\right)\Big|, \qquad \delta=+,-. \end{aligned} \tag{6.29}$$

Now the assumption (6.24) entails

$$
\lim_{t\to\infty} \eta^\delta(t) = \begin{vmatrix} M_{11}^\delta & \cdots & M_{1n}^\delta & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{n1}^\delta & \cdots & M_{nn}^\delta & 0 & \cdots & 0 \\ M_{n+1,1}^\delta & \cdots & M_{n+1,n}^\delta & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ M_{N1}^\delta & \cdots & M_{Nn}^\delta & 0 & \cdots & 1 \end{vmatrix} = \left| M_n^\delta \right|, \tag{6.30}
$$

whence (6.25) is clear. The assumption (6.26) gives rise to the same limit matrix as in (6.30), except that one should add $\left(M_{1,n+1}^\delta, \ldots, M_{N,n+1}^\delta \right)^t$ to the $(n+1)$th column. A moment's thought then yields

$$
\lim_{t\to\infty} \eta^\delta(t) = \left| M_n^\delta \right| + \left| M_{n+1}^\delta \right|, \tag{6.31}
$$

and so (6.27) results.   ∎

Proof of Proposition 6.1. We only show

$$
\lim_{t\to\infty} \frac{\left| \mathbf{1}_N + L(x_0 + s_0 t, t) U^{-1} \right|}{\left| \mathbf{1}_N + L(x_0 + s_0 t, t) U \right|} = \exp\left(-i F^{(+)}(x_0, s_0) \right), \tag{6.32}
$$

the proof for $t \to -\infty$ being similar. Let us first note that the ordering of the positions entails

$$
-v(\alpha_{N_-}^-) < \cdots < -v(\alpha_1^-) < v(\alpha_{N_+}^+) < \cdots < v(\alpha_1^+). \tag{6.33}
$$

For $s_0 > v(\alpha_1^+)$ we therefore have $\lambda_i(s_0) < 0$, $i = 1, \ldots, N$, cf. (6.20). Thus we can invoke Lemma 6.2 with

$$
M^\pm = L(x_0, 0) U^{\mp 1}, \tag{6.34}
$$

and $n = 0$ in (6.24) to obtain limit 1 on the lhs of (6.32). Since we have

$$
F^{(+)}(x_0, s_0) = 0, \qquad s_0 > v(\alpha_1^+), \tag{6.35}
$$

we obtain (6.32) for $s_0 > v(\alpha_1^+)$.

For $s_0 \in (v(\alpha_2^+), v(\alpha_1^+))$ we have $\lambda_1(s_0) > 0$, with all other $\lambda_i(s_0)$ still negative. Thus Lemma 6.2 applies with (6.34) in effect, and with $n = 1$ in (6.24). Then the lhs of (6.32) yields $\exp(2i\alpha_1^+)$, which equals the rhs (recall (6.18)). Likewise, it follows that (6.32) holds true for s_0 not equal to the velocities (6.33).

Next, we choose $s_0 = v(\alpha_1^+)$. Then we have $\lambda_1(s_0) = 0$, with all other $\lambda_i(s_0)$ negative. Thus we can exploit Lemma 6.2 with $n = 0$ in (6.26), which yields a limit

$$
\frac{1 + (L(x_0, 0) U^{-1})_{11}}{1 + (L(x_0, 0) U)_{11}} = \frac{1 + \exp(\theta_1^+ - 2\alpha_1^+ x_0) V_1^+ \exp(i\alpha_1^+)}{1 + \exp(\theta_1^+ - 2\alpha_1^+ x_0) V_1^+ \exp(-i\alpha_1^+)}. \tag{6.36}
$$

This agrees with the rhs of (6.32), cf. (5.8), (6.9) and (6.10).

Choosing now $s_0 = v(\alpha_2^+)$, Lemma 6.2 with $n = 1$ in (6.26) yields the limit

$$
\exp(2i\alpha_1^+) \cdot \frac{1 + m_2 \exp(i\alpha_2^+)}{1 + m_2 \exp(-i\alpha_2^+)}, \tag{6.37}
$$

where m_2 denotes the quotient of the principal minor of $L(x_0, 0)$ with respect to indices 1, 2 and its 11-element. Cauchy's identity yields

$$m_2 = \exp(\theta_2^+ - 2\alpha_2^+ x_0) V_2^+ \tanh^2[(q_1^+ - q_2^+)/2], \tag{6.38}$$

and so the limit agrees with the rhs, viz.,

$$\exp\left[2i\alpha_1^+ - if^+(q_2^+, \theta_2^+; x_0 + \Delta_2^+(q^+, q^-)/4\alpha_2^+)\right], \tag{6.39}$$

cf. (6.9), (6.10), and (6.18).

Proceeding in the same way for s_0 equal to the remaining velocities, it is readily checked that (6.32) holds true as well. Therefore, Proposition 6.1 now follows. ■

Finally, we briefly consider the above N-soliton solutions from the perspective of Section 7 in Ref. [5], especially as concerns the issue of soliton space-time trajectories. Inspecting the derivatives (2.55) and (2.60), we see that there is an obvious choice of space-time trajectories for the 1-soliton solutions, namely,

$$x^+(t) = \theta^+/2\alpha^+ + v(\alpha^+)t, \tag{6.40}$$

$$x^-(t) = \theta^-/2\alpha^- - v(\alpha^-)t. \tag{6.41}$$

Let us now choose $N_+ = N$, $N_- = 0$ until further notice. Then we are dealing with the self-dual $II_{rel}(\tau = \pi/2)$ system. For the soliton solutions arising in Section 7 of Ref. [5] this specialization amounts to the pure soliton case studied in Subsection 7A. In order to compare the present setting to *loc. cit.*, we should take $\beta = 1$, $\tau = \pi/2$ in *loc. cit.*, and substitute $q, \theta \to \hat{\theta}, \hat{q}$. Then the above Lax matrix (5.2) turns into a diagonal similarity transform of the Lax matrix (7.4) in *loc. cit.*

Substituting next in *loc. cit.*

$$y \to x, \qquad \sigma_j = 2\alpha_j^+, \qquad v_j = v(\alpha_j^+), \tag{6.42}$$

one gets agreement with Eq. (7.7). Since $L(x, t)$ (6.2) amounts to a diagonal similarity transform of the matrix $\tilde{A}(t, x)$ (7.6), we may apply Theorem 7.1 to the case at hand. Thus we obtain non-intersecting soliton space-time trajectories with long-time asymptotics

$$x^+_{\substack{j \\ N-j+1}}(t) = \frac{1}{2\alpha_j^+}\left(\theta_j^+ \mp \frac{1}{2}\Delta_j^+(q^+)\right) + v(\alpha_j^+)t + O(\exp(\mp tr_j)), \qquad t \to \pm\infty, \tag{6.43}$$

where

$$\Delta_j^+(q^+) \equiv \left(\sum_{k<j} - \sum_{k>j}\right)\ln\left(\coth^2[(q_j^+ - q_k^+)/2]\right), \quad qquadj = 1, \ldots, N, \tag{6.44}$$

$$r_j \equiv \min_{k\neq j} 2\alpha_k^+ |v(\alpha_k^+) - v(\alpha_j^+)|, \qquad j = 1, \ldots, N. \tag{6.45}$$

(See Ref. [20] for a picture of sine-Gordon soliton space-time trajectories.)

To be sure, without prior knowledge it would not at all be obvious that the terminology used here is appropriate for the solution (6.4). Indeed, since $L(x, t)$ is multiplied

by the q^+-dependent phase matrices U and U^*, we have no analog of the "1-particle superposition" formulas Eqs. (7.10)–(7.14) in *loc. cit.*. More importantly, at face value the long-time asymptotics of the spectrum of $L(x,t)$ seems to have no bearing on the long-time asymptotics of the solution $\Psi(x,t)$ (6.4).

Even so, we need only invoke Proposition 6.1 for the case $N_+ = N$, $N_- = 0$ at hand to deduce that the asymptotic space-time trajectories in (6.43) coincide with the 1-soliton trajectories in (6.9), cf. also (6.40). Accordingly, the trajectories in (6.43) do encode the physical characteristics of the right-moving N-soliton solutions.

Likewise, for the case $N_+ = 0$, $N_- = N$ we can use Theorem 7.1 in *loc. cit.* to obtain non-intersecting space-time trajectories that coincide with the obvious "locations" of the N left-moving solitons for asymptotic times.

Whenever $N_+ N_- > 0$, however, Theorem 7.1 can no longer be used to define space-time trajectories. This is because $L(x,t)$ need not have positive and simple spectrum when particles and antiparticles are present. Indeed, as $(q^+, q^-, \theta^+, \theta^-)$ varies over Ω (5.39), the eigenvalues of $L(0,0)$ range over all of the right half plane, cf. *loc. cit.*, Subsection 2C.

As an illuminating example, consider the case $N_+, N_- = 1$ with $q \equiv (q^+ - q^-)/2$:

$$
L(x,t) = \begin{pmatrix} \tanh|q| & i/\cosh q \\ i/\cosh q & \tanh|q| \end{pmatrix} \cdot \begin{pmatrix} \exp(\theta^+(x,t)) & 0 \\ 0 & \exp(\theta^-(x,t)) \end{pmatrix}, \tag{6.46}
$$

$$
\theta^\delta(x,t) \equiv \theta^\delta - 2\alpha^\delta \left(x - \delta v(\alpha^\delta)t \right), \qquad \delta = +, -. \tag{6.47}
$$

Setting

$$
\sigma(x,t) \equiv \tanh^2([\theta^+(x,t) - \theta^-(x,t)]/2) - 1/\cosh^2 q, \tag{6.48}
$$

it is routine to verify that for $\alpha^+ \neq \alpha^-$ the matrix $L(x,t)$ has two distinct positive eigenvalues when $\sigma(x,t) > 0$ and a complex-conjugate pair of eigenvalues in the right half plane when $\sigma(x,t) < 0$, whereas $L(x,t)$ is not diagonalizable when $\sigma(x,t)$ vanishes. For an *arbitrary* fixed t_0, one therefore finds that $L(x,t_0)$ has two distinct positive eigenvalues for $x > x_0^+$ and $x < x_0^-$, and non-real spectrum for $x \in (x_0^-, x_0^+)$. Moreover, in the non-generic case $\alpha^+ = \alpha^-$ the spectral character only depends on t_0. (As a bonus, these observations show that the Hamiltonians on Ω that generate the x- and t-flows generically do not leave the spectral decomposition of Ω invariant.)

In spite of this quite different state of affairs for $N_+ N_- > 0$, we believe that a more refined spectral analysis of $L(x,t)$ should yield N non-intersecting space-time trajectories for $t > T^{(+)}$ and $t < T^{(-)}$, where $T^{(\pm)}$ depend only on $(q,\theta) \in \Omega$ (5.39). The asymptotics of these trajectories should read

$$
x^+_{\substack{j \\ N_+-j+1}}(t) \sim \frac{1}{2\alpha_j^+} \left(\theta_j^+ \mp \frac{1}{2}\Delta_j^+(q^+,q^-) \right) + v(\alpha_j^+)t, \qquad t \to \pm\infty, \tag{6.49}
$$

$$
x^-_{\substack{l \\ N_--l+1}}(t) \sim \frac{1}{2\alpha_l^-} \left(\theta_l^- \mp \frac{1}{2}\Delta_l^-(q^+,q^-) \right) - v(\alpha_l^-)t, \qquad t \to \pm\infty, \tag{6.50}
$$

(with Δ_j^+ and Δ_l^- given by (6.10) and (6.11)), so that the trajectories coincide asymptotically with the 1-soliton trajectories following from (6.9).

To explain what is involved here, it is illuminating to reconsider the example (6.46)–(6.48) with $q^+ = -q^-$ (so that $\alpha^+ = \alpha^-$). Then one readily verifies the following features. First, there exist unique $T^{(+)}$, $T^{(-)}$ such that $L(x,t)$ has non-real spectrum for $t \in (T^{(-)}, T^{(+)})$ and distinct positive eigenvalues for $t > T^{(+)}$ and $t < T^{(-)}$. Second, for $t > T^{(+)}$ there exist unique $x^+(t)$, $x^-(t)$ with $x^+(t) > x^-(t)$ such that $L(x^\delta(t),t)$, $\delta = +,-$, has non-degenerate eigenvalue 1. Likewise, for $t < T^{(-)}$ there exist unique $x^+(t)$, $x^-(t)$ with $x^+(t) < x^-(t)$ such that $L(x^\delta(t),t)$, $\delta = +,-$, has non-degenerate eigenvalue 1. Finally, the asymptotics of the trajectories $x^\delta(t)$, $\delta = +,-$, is given by (6.49) and (6.50). (Physically speaking, the particle and antiparticle associated with this special two-soliton solution form a virtual bound state/resonance for $t \in [T^{(-)}, T^{(+)}]$.)

Of course, this example may be viewed as too special to yield convincing evidence. Indeed, a far more important reason why it is plausible that trajectories with the above-mentioned properties exist is the following lemma.

Lemma 6.3. *Let $M \in \mathcal{M}$ (6.21) and let*

$$E(t) \equiv Me^{tD}, \tag{6.51}$$

where D is given by (6.23) and (6.26). Then there exists $T \in \mathbb{R}$ such that for all $t \geq T$ the matrix $E(t)$ has a non-degenerate eigenvalue $e_n(t)$ obeying

$$e_n(t) = m_n(1 + \rho_n(t)), \qquad m_n \equiv |M_n|/M_{n-1}|, \tag{6.52}$$

$$|\rho_n(t)| \leq C \exp(-tr_n), \qquad r_n \equiv \min(d_1,\ldots,d_n,-d_{n+2},\ldots,-d_N). \tag{6.53}$$

Proof. This lemma can be obtained as a corollary of the proof of Theorem A2 in Ref. [21]. The latter theorem concerns the case where $d_1,\ldots,d_N$ are distinct, which is not implied by our assumption (6.26). But when one follows the arguments in its proof, one easily sees that they entail the assertion of the lemma. ∎

We now explain the bearing of this lemma on our trajectory conjecture. To this end we consider (for example) the matrix

$$L(x_0 + v(\alpha_j^+)t, t) = L(x_0, 0) \operatorname{diag}\left(\exp(\lambda_1(v(\alpha_j^+))t),\ldots,\exp(\lambda_N(v(\alpha_j^+))t)\right), \tag{6.54}$$

where we used (6.2) and (6.20). Taking $M = L(x_0, 0)$ and $n = j$ in Lemma 6.3, we see that the assumptions are satisfied. Since the linear functions $\lambda_i(s)$ (6.20) intersect for s varying over a finite set $\mathcal{I}$, the numbers $d_1,\ldots,d_N$ in the lemma are not distinct whenever $v(\alpha_j^+) \in \mathcal{I}$. This is the reason why Theorem A2 in Ref. [21] does not apply in general. (For s-values not in $\mathcal{I}$, however, Theorem A2 does apply to $L(x_0 + st, t)$, entailing simple and positive spectrum for $|t|$ sufficiently large.)

As a consequence, we deduce that for $t > T_j$ the matrix (6.54) has a non-degenerate eigenvalue $e_j(x_0, t)$. The trajectory $x_j^+(t)$ should now be defined so that

$$e_j(x_j^+(t) - v(\alpha_j^+)t, t) = 1. \tag{6.55}$$

One readily checks that this would yield the desired asymptotics for $t \to \infty$, but we cannot rigorously prove that there exists a unique $x_0 \in \mathbb{R}$ such that $e_j(x_0, t) = 1$. (For one thing, the choice of T_j *depends* on the x_0 we fix.)

In any event, by now it will be clear what spectral feature our trajectory conjecture amounts to: For large times t, the matrix $L(x, t)$ should have a non-degenerate eigenvalue 1 for N and only N positions $x_{N_-}^-(t) < \cdots < x_1^-(t) < x_{N_+}^+(t) < \cdots < x_1^+(t)$ — the soliton space-time trajectories. Though the above arguments do not constitute a complete proof, they provide considerable evidence. To conclude, it should be emphasized that for $N_+ N_- > 0$ one should not expect global trajectories. Indeed, as we have already established explicitly for the above example (6.46) with $q^- = -q^+$, it is likely that when left- and right-moving trajectories collide, the corresponding eigenvalue pair becomes non-real.

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