Journal of Nonlinear Mathematical Physics Volume 8, Number 2 (2001), 196–210 LETTER

# Invariants of PL Manifolds from Metrized Simplicial Complexes. Three-Dimensional Case

*Igor G KOREPANOV*

*South Ural State University, 76 Lenin ave., Chelyabinsk 454080, Russia*
*E-mail: kig@susu.ac.ru*

*Received October 6, 2000; Revised January 16, 2001; Accepted January 17, 2001*

**Abstract**

An invariant of three-dimensional orientable manifolds is built on the base of a solution of pentagon equation expressed in terms of metric characteristics of Euclidean tetrahedra.

## Introduction

The motivation for work presented in this Letter is, from the side of mathematical physics, in developing some ideas concerning building of a topological field theory from a variant of "Regge calculus". As is well-known, Regge [1] proposed a discretization of space-time in the form of its triangulation and assigning lengths to the edges of such triangulation. In order to construct an analog of functional integral in such theory, either a sum is taken over "all" triangulations or some sort of equivalence of different triangulations is established.

The idea of such equivalence is well known in pure mathematics. Mathematically, we want to put some numeric characteristics in correspondence to a PL (piecewise-linear) manifold. The usual way of doing this is, roughly speaking, as follows. First, describe the manifold in algebraic or combinatorial terms — we have already done it as soon as we have chosen a triangulation. Such a description can be done, as a rule, in numerous different ways, but those different ways can often be obtained from one another by using a sequence of some simple re-buildings, or "moves". In the case of triangulation, such moves affect only a few neighboring simplices from their total maybe very big number.

Then, one could try to find an algebraic expression which could be put in correspondence to some "local" part of the manifold, e.g., to a cluster of neighboring simplices, such that it would remain similar to itself in some sense under the mentioned moves. Finally, one could try to construct a "global" expression out of the "local" ones, in conformity with their algebraic structure, and find some way to extract manifold invariants from such an expression.

A typical case of realization of the stated program is the building of three-dimensional manifold invariants out of quantum $6j$-symbols. As key property of $6j$-symbols one can take the fact that they satisfy the *pentagon equation* which is depicted in a natural way as the equality of two diagrams, the first containing two tetrahedra with a common

*Copyright © 2001 by I G Korepanov*

base, while the second — three tetrahedra occupying the same domain in a Euclidean space, see Figure 1 below. Diagrams of such sort can be also introduced for the space of any dimension $n$ (the left- and right-hand sides must form together, at least from the combinatorial viewpoint, the boundary of an $(n+1)$-simplex). It seems however that any direct analogs of quantum $6j$-symbols for higher-dimensional manifolds are rather hard to find.

We would like to propose some other algebraic expressions that obey a relation which, too, deserves the name of pentagon equation, because the picture for it is the same. In constructing our expressions we assume that the tetrahedra lie in a usual Euclidean space and thus possess metric characteristics such as edge lengths, dihedral angles and volumes. Our invariant is a certain expression made of those values. Thus, it may be thought of as produced by some version of Regge calculus. In order that our invariant be well defined, we will assume that our PL manifold satisfies some additional requirements including *orientability*.

The experience of the theory of discrete integrable models shows that equations that are depicted by the same diagram turn out ultimately to be closely connected, even if the diagram seems at first to have completely different meanings. Besides, a connection of our expressions with usual $6j$-symbols is suggested by Justin Roberts' work [3] where he explains how a metric tetrahedron appears in the quasiclassical limit from $6j$-symbols corresponding to $SU(2)$ group (this was first discovered by Ponzano and Regge [2], but not proved rigorously). Here the quasiclassics is understood as tending of the irreducible representations' dimensions to infinity. In this sense, our invariant looks "quasiclassical" too, but it is worth mentioning that, again, the theory of integrable models teaches that the relations between quantum and classical models are much richer than just classical models being a limiting case of quantum ones, and in fact quantum models are sometimes studied using "classical" considerations.

It looks plausible that our constructions can be generalized to higher-dimensional manifolds. Thus, the aim of the present Letter is not only in introducing still more invariants of three-dimensional manifolds but in elaborating the necessary technical devices, starting from this simplest case.

Below, in Section 1 we recall the derivation of the "local" formula from paper [4]. This formula contains the partial derivative of "defect angle" around an edge common for three tetrahedra in the length of that edge, taken in the neighborhood of the flat case (when the whole cluster of tetrahedra can be imbedded into a 3-dimensional Euclidean space). Generalization (globalization) of this formula onto the case of simplicial complexes having *many* tetrahedra requires some technical work and occupies Sections 2–5.

In Section 2 we introduce a matrix $A$ of partial derivatives of *all* defect angles (we call them simply "curvatures") in *all* edge lengths. Somewhat unexpectedly, this matrix turns out to be *symmetric*: $A = A^{\mathrm{T}}$. In Section 3 we investigate matrix $A$ from another point of view: it is strongly degenerate, and we get first results in globalizing the formula for $2 \to 3$ (2 tetrahedra to 3 tetrahedra) moves from Section 1 by using matrix $A$'s minor of the highest rank. It is also in Section 3 that we begin using the orientability of the manifold. In Section 4 we continue our technical work and construct a differential form not depending on the choice of the mentioned minor and behaving very nicely under $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ moves.

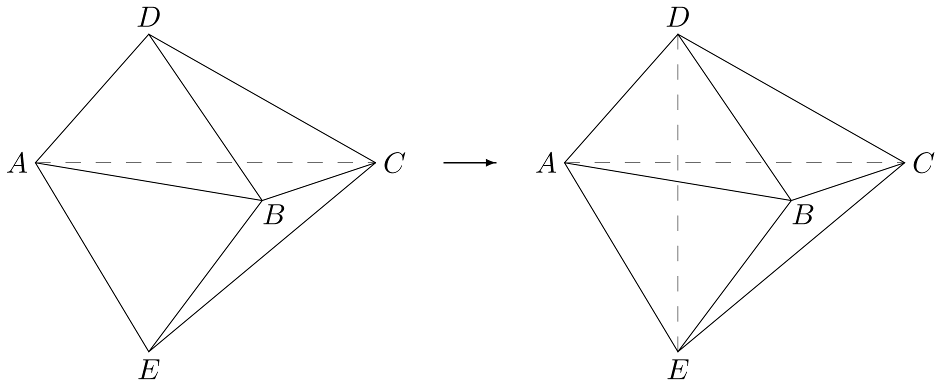


**Figure 1.** A 2 → 3 Pachner move

In Section 5 we divide this differential form by some "standard" differential form and obtain a number that does not depend on any edge length! Thus the explicit formula appears for the invariant of a three-manifold.

To demonstrate the efficiency of our formula, we calculate in Section 6 our invariant in the two simplest examples, namely, for the sphere $S^3$ and the projective space $RP^3$. Finally, we discuss our results and their possible generalizations in Section 7.

## 1 The local formula

In this section we recall the derivation of the formula from [4] that can be treated as a sort of pentagon equation involving five tetrahedra in a three-dimensional space. It corresponds in a natural way to replacing a cluster of two Euclidean tetrahedra with the cluster of three tetrahedra that covers the same 3-domain, or a 2 → 3 *Pachner move*, as in Figure 1.

Consider five points $A$, $B$, $C$, $D$ and $E$ in the three-dimensional Euclidean space. There exist ten distances between them, which we will denote as $l_{AB}$, $l_{AC}$ and so on.

Let us fix all the distances except $l_{AB}$ and $l_{DE}$. Then, $l_{AB}$ and $l_{DE}$ satisfy one constraint (Cayley–Menger equation) which we can, using arguments like those in [5], represent in the following differential form:

$$\left| \frac{l_{AB}\, dl_{AB}}{V_{\overline{D}}\, V_{\overline{E}}} \right| = \left| \frac{l_{DE}\, dl_{DE}}{V_{\overline{A}}\, V_{\overline{B}}} \right|, \tag{1}$$

where, say, $V_{\overline{A}}$ denotes the volume of tetrahedron $\overline{A}$, that is one with vertices $B$, $C$, $D$ and $E$ (and *without* $A$).

Let us consider the dihedral angles at the edge $DE$ — the common edge for tetrahedra $\overline{A}$, $\overline{B}$ and $\overline{C}$. Namely, denote

$$\angle BDEC \stackrel{\text{def}}{=} \alpha, \qquad \angle CDEA \stackrel{\text{def}}{=} \beta, \qquad \angle ADEB \stackrel{\text{def}}{=} \gamma.$$

We have:

$$0 = d(\alpha + \beta + \gamma) = \frac{\partial \gamma}{\partial l_{AB}}\, dl_{AB} + \frac{\partial(\alpha + \beta + \gamma)}{\partial l_{DE}}\, dl_{DE}. \tag{2}$$

According to [5, formula (11)],

$$\left|\frac{\partial\gamma}{\partial l_{AB}}\right| = \frac{1}{6}\left|\frac{l_{AB}\, l_{DE}}{V_{\overline{C}}}\right|. \tag{3}$$

Denote also

$$\alpha+\beta+\gamma \stackrel{\text{def}}{=} 2\pi - \omega_{DE}, \tag{4}$$

where $\omega_{DE}$ is the "defect angle" around edge $DE$. The formulas (1)–(4) together yield

$$\left|\frac{1}{V_{\overline{D}}\, V_{\overline{E}}}\right| = \frac{1}{6}\left|\frac{l_{DE}^2}{V_{\overline{A}}\, V_{\overline{B}}\, V_{\overline{C}}}\left(\frac{\partial\omega_{DE}}{\partial l_{DE}}\right)^{-1}\right|. \tag{5}$$

**Remark.** This can be also written by means of the following integral in the length of the edge $DE$ "redundant" for the tetrahedra $\overline{D}$ and $\overline{E}$:

$$\left|\frac{1}{V_{\overline{D}}\, V_{\overline{E}}}\right| = \frac{1}{6}\left|\int \frac{\delta(\omega_{DE})\, l_{DE}^2\, dl_{DE}}{V_{\overline{A}}\, V_{\overline{B}}\, V_{\overline{C}}}\right|, \tag{6}$$

with the integral taken over a neighborhood of the value of $l_{DE}$ corresponding to the flat space ($\omega_{DE}=0$); $\delta$ is the Dirac delta function. However, the straightforward attempt to globalize formula (6) runs into diverging integrals, and the right "global" formulae (see, e.g., (30)) will have volumes raised into the power $(-1/2)$ rather than $(-1)$.

## 2 Reciprocity theorems for lengths and defect angles

In this Section we will do some of the technical work mentioned in the Introduction. Consider a finite simplicial complex made of tetrahedra and their faces (of dimensions 2, 1 and 0). If the contrary is not stated explicitely, we assume that every 2-face belongs to boundaries of exactly two tetrahedra lying at its different sides (thus, the corresponding PL-manifold as a whole has no boundary).

Assign to each edge of the complex a *length*, say length $l_a$ to the edge $a$. Consider a cluster of all tetrahedra containing the edge $a$. The edge lengths in this cluster may happen to be consistent in such way that the whole cluster can be put into a Euclidean 3-space. Generally, however, there is an obstacle called the *defect angle* $\omega_a$ corresponding to edge $a$ which we define *up to a multiple of* $2\pi$ by the equality

$$\omega_a \equiv -\alpha-\beta-\cdots-\eta \pmod{2\pi},$$

where $\alpha,\beta,\ldots,\eta$ are the proper dihedral angles of the tetrahedra.

Now we will consider the partial derivatives like $\partial\omega_a/\partial l_b$ which are taken with the fixed lengths of all edges except $b$. It must be clear from the preceding paragraph that such a partial derivative may be nonzero only if the edges $a$ and $b$ belong to a single tetrahedron.

**Theorem 1 (local reciprocity theorem).** *Let a tetrahedron in the Euclidean space be given, $a$ and $b$ being its two edges (they can lie on skew, intersecting or coinciding straight lines), $l_a$ and $l_b$ being their lengths, and $\varphi_a$ and $\varphi_b$ — dihedral angles at those edges. Then*

$$\frac{\partial\varphi_a}{\partial l_b} = \frac{\partial\varphi_b}{\partial l_a}. \tag{7}$$

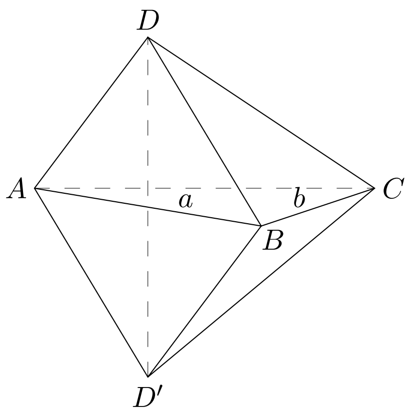


**Figure 2.** To the proof of Theorem 1

**Proof.** The case of coinciding edges $a$ and $b$ is trivial.

The case of skew edges: both l.h.s. and r.h.s. of (7) equal $(1/6)l_a l_b/V$, where $V$ is the volume of tetrahedron (compare formula (3)).

The case of intersecting edges. Let $a$ and $b$ be, respectively, edges $AB$ and $BC$ in a tetrahedron $ABCD$. Consider also the mirror image $ABCD'$ of tetrahedron $ABCD$ with respect to the plane $ABC$, see Figure 2.

Now we can calculate, say, $\partial\varphi_b/\partial l_a$ in the following way. Assuming that all the edge lengths except $a$ and $DD'$ in Figure 2 are fixed, let us calculate first $\partial l_{DD'}/\partial l_a$. We will get, similarly to formula (1) (and using the same notations like $V_{\overline{A}}$):

$$\frac{\partial l_{DD'}}{\partial l_a} = -\frac{l_a}{l_{DD'}} \frac{V_{\overline{A}} V_{\overline{B}}}{V_{ABCD}^2}. \tag{8}$$

Next, from tetrahedron $BCDD'$ (in other words — tetrahedron $\overline{A}$) we can get (compare formula (3)):

$$2\frac{\partial\varphi_b}{\partial l_{DD'}} = \frac{1}{6}\frac{l_b l_{DD'}}{V_{\overline{A}}}. \tag{9}$$

It follows from (8) and (9) that

$$\frac{\partial\varphi_b}{\partial l_a} = -\frac{1}{12}\frac{l_a l_b V_{\overline{B}}}{V_{ABCD}^2}.$$

Clearly, the result will be the same for $\partial\varphi_a/\partial l_b$. The theorem is proved.

**Theorem 2 (global reciprocity theorem).** *Let a complex be given of the type described in the beginning of this Section. Select in it two edges $a$ and $b$. Then*

$$\left.\frac{\partial\omega_a}{\partial l_b}\right|_{l_c \text{ are constant for } c\neq b} = \left.\frac{\partial\omega_b}{\partial l_a}\right|_{l_c \text{ are constant for } c\neq a}. \tag{10}$$

**Proof.** The equality (10) follows from the fact that the l.h.s. of (10) is the sum of values of the type $-\partial\varphi_a^{(k)}/\partial l_b$, where $k$ numbers the tetrahedra containing *both edges* $a$ and $b$, and $\varphi_a^{(k)}$ is the dihedral angle in such tetrahedron at edge $a$. As for the r.h.s. of (10), it is the sum of similar terms but with interchanged $a \leftrightarrow b$, and these sums are equal due to the local reciprocity theorem. The theorem is proved.

**Addition to Theorem 2.** The equality (10) remains valid if we change the definitions of defect angles in the following way: select any subset in the set of tetrahedra of the complex, and assume *all* dihedral angles in those tetrahedra to be *negative*.

**Proof** follows immediately from an obvious modification of the proof of Theorem 2.

Introduce the matrix

$$A = \left( \frac{\partial \omega_j}{\partial l_k} \right), \qquad (11)$$

where $j$ and $k$ run through all the edges of the complex. Matrix $A$ is thus *symmetric*: $A = A^{\mathrm{T}}$.

## 3 A quantity good for 2 → 3 moves

Throughout the rest of this Letter, we will be considering "metrized" simplicial 3-complexes (with lengths assigned to their edges) of the type described in the beginning of Section 2 with the following two additional constraints: *the corresponding PL manifold must be orientable, and the lengths are such that the corresponding polyhedron can be put into the 3-dimensional Euclidean space* $R^3$. This will be understood as follows: we identify all *vertices* of the complex with points in $R^3$. Thus, all edges acquire Euclidean lengths, and every tetrahedron gets embedded in $R^3$. It is important to note that we do permit any self-intersections of the obtained Euclidean tetrahedra.

We say that such lengths form a *permitted length configuration*. At the same time, we will consider *any infinitesimal deformations* of lengths which can thus draw the complex out of the Euclidean space or, in other words, produce some infinitesimal defect angles around edges. For brevity, we will sometimes call those defect angles "curvatures" (as an exception from this rule, we will soon be considering a situation where *one* "new" edge can take any value, but if we remove it the complex fits again into Euclidean space).

The accurate definition of infinitesimal defect angles shows why we require the orientability of the manifold. Fix a consistent orientation of all tetrahedra in the complex. When we map the complex into a Euclidean space (which we suppose to have its own fixed orientation), some of the tetrahedra preserve their orientation while the others change it. We will define the defect angle around a given edge as the algebraic sum of (interior) dihedral angles in adjoining tetrahedra taken with the sign $-$ for the tetrahedra that do not change their orientation and with the sign $+$ for the rest of them. Such definition ensures that the defect angles in a complex mapped into Euclidean space will be zero (in absense of infinitesimal deformations), and we will be using the Addition to Theorem 2 exactly in such situation.

The matrix $A$ given by (11) is usually strongly degenerate, see examples below in Section 6. It follows from the fact that $A$ is symmetric and standard theorems in linear algebra that there exists a *diagonal* nondegenerate submatrix $A|_{\mathcal{C}}$ of $A$ of sizes rank $A \times$ rank $A$. This means that we can choose a subset $\mathcal{C}$ in the set of all edges, and leave only those rows and columns in $A$ that correspond (both rows and columns) to edges from $\mathcal{C}$.

Now we will make the considerations of the preceding paragraph more precise in the following way. Matrix $A$ depends on a chosen permitted length configuration. We will be dealing with the ranks of $A$ and its submatrices for a *generic* permitted configuration. Accordingly, below we denote by $\mathcal{C}$ a chosen subset of $L$ edges for which $A|_{\mathcal{C}}$ is nondegenerate in the general position, where $L$ equals rank $A$ again in the general position.

The rest of edges form the subset that we will denote $\overline{\mathcal{C}}$.

**Lemma 1.** *The form* $\bigwedge_{i\in\overline{\mathcal{C}}} dl_i$*, i.e. the exterior product of differentials of all edge lengths from* $\overline{\mathcal{C}}$*, is nondegenerate in a generic point of the algebraic variety consisting of all permitted length configurations.*

**Proof.** The lemma can be reformulated as follows: for any set of length differentials $dl_i$ of edges in $\overline{\mathcal{C}}$ one can find such length differentials of edges in $\mathcal{C}$ that *all* infinitesimal curvatures $d\omega$ will equal zero. Now, it follows immediately from the nondegeneracy of matrix $A|_{\mathcal{C}} = (\partial\omega_j/\partial l_k)|_{\mathcal{C}}$ that one can always find such differentials of lengths in $\mathcal{C}$ that all the infinitesimal curvatures *around edges in* $\mathcal{C}$ will be zero. But any other infinitesimal curvature *is linearly dependent* upon the curvatures in $\mathcal{C}$ and thus vanishes as well. The lemma is proved.

Consider two adjacent tetrahedra (with a common 2-face) in the complex and perform the operation of replacing them with three tetrahedra, as in Section 1. In doing so, we add a new edge ($DE$ in Figure 1) of length $l_{\text{new}}$. Denote $\tilde{l}_{\text{new}} \stackrel{\text{def}}{=} l_{\text{new}} - l_{\text{new}}^{(0)}$, where $l_{\text{new}}^{(0)}$ is such value of $l_{\text{new}}$ where the curvature $\omega_{\text{new}}$ around the new edge is exactly zero. Then

$$d\tilde{l}_{\text{new}} = dl_{\text{new}} - a_1\, dl_1 - \cdots - a_N\, dl_N, \tag{12}$$

where $N$ is the number of edges before adding the new one, and

$$a_k = \frac{\partial l_{\text{new}}^{(0)}}{\partial l_k} = \left.\frac{\partial l_{\text{new}}}{\partial l_k}\right|_{\substack{\omega_{\text{new}}=0\\ dl_1=\cdots=dl_N=0}} = -\frac{\partial\omega_{\text{new}}/\partial l_k}{\partial\omega_{\text{new}}/\partial l_{\text{new}}}.$$

It is clear that $d\omega_{\text{new}}$ depends only on $d\tilde{l}_{\text{new}}$ and does not depend on $dl_1, \ldots, dl_N$:

$$d\omega_{\text{new}} = \frac{\partial\omega_{\text{new}}}{\partial l_{\text{new}}}\, d\tilde{l}_{\text{new}}, \tag{13}$$

where we have taken into account the obvious equality

$$\left.\frac{\partial}{\partial\tilde{l}_{\text{new}}}\right|_{dl_1=\cdots=dl_N=0} = \left.\frac{\partial}{\partial l_{\text{new}}}\right|_{dl_1=\cdots=dl_N=0}. \tag{14}$$

Then, when $d\tilde{l}_{\text{new}} = 0$, the definition of matrix $A$ works:

$$\begin{pmatrix} d\omega_1 \\ \vdots \\ d\omega_N \end{pmatrix} = A \begin{pmatrix} dl_1 \\ \vdots \\ dl_N \end{pmatrix}. \tag{15}$$

Combining (13), (14) and (15) we can write:

$$\begin{pmatrix} d\omega_{\text{new}} \\ d\omega_1 \\ \vdots \\ d\omega_N \end{pmatrix} = \begin{pmatrix} \partial\omega_{\text{new}}/\partial l_{\text{new}} & 0 \cdots 0 \\ \partial\omega_1/\partial l_{\text{new}} & \\ \vdots & A \\ \partial\omega_N/\partial l_{\text{new}} & \end{pmatrix} \begin{pmatrix} d\tilde{l}_{\text{new}} \\ dl_1 \\ \vdots \\ dl_N \end{pmatrix}. \tag{16}$$

We are willing to construct the new matrix $A_{\text{new}}$ of sizes $(N+1)\times(N+1)$ that will link, like matrix $A$ did, the differentials of lengths and curvatures, but for the complex with the added edge. Combining (12) and (16) we get:

$$A_{\text{new}} = \begin{pmatrix} \partial\omega_{\text{new}}/\partial l_{\text{new}} & 0 \cdots 0 \\ \partial\omega_1/\partial l_{\text{new}} & \\ \vdots & A \\ \partial\omega_N/\partial l_{\text{new}} & \end{pmatrix} \begin{pmatrix} 1 & -a_1 & \cdots & -a_N \\ & 1 & & \\ & & \ddots & 0 \\ & 0 & & 1 \end{pmatrix}. \tag{17}$$

Let $\mathcal{C}_{\text{new}}$ be the subset of the set of edges obtained by adding the "new" edge to $\mathcal{C}$. It can be seen from formula (17) that we can take $A_{\text{new}}|_{\mathcal{C}_{\text{new}}}$ for a submatrix of $A_{\text{new}}$ having the same rank as $A_{\text{new}}$. To be exact, it follows from (17) that

$$\det\left(A_{\text{new}}|_{\mathcal{C}_{\text{new}}}\right) = \frac{\partial\omega_{\text{new}}}{\partial l_{\text{new}}} \det\left(A|_{\mathcal{C}}\right). \tag{18}$$

Note that $\partial\omega_{\text{new}}/\partial l_{\text{new}}$ has been calculated in Section 1, see formula (5). With this taken into account, formula (18) shows that the following theorem is valid.

**Theorem 3.** *The expression*

$$\frac{f}{\prod\limits_{\text{over all edges}} l^2} \prod_{\text{over all tetrahedra}} 6V, \tag{19}$$

*where*

$$f \stackrel{\text{def}}{=} \det\left(A|_{\mathcal{C}}\right), \tag{20}$$

*and $l$'s and $V$'s are of course lengths and volumes, does not change or changes only its sign under performing a $2 \to 3$ Pachner move in such way that a new edge is added to the subset $\mathcal{C}$.*

In order to construct out of (19) an invariant of a PL manifold, we still have to get rid of the dependence of our construction on the concrete choice of subset $\mathcal{C}$ and also make our formulas describe not only $2 \leftrightarrow 3$ moves (adding or removing an edge, as in Section 1), but also $1 \leftrightarrow 4$, when a new *vertex* is added to the complex or removed. This is what we will do next.

## 4 Differential forms and Pachner moves

In this Section we are going to consider the following column vectors of differentials:

- $\boldsymbol{dl}$ — the column of differentials of lengths for edges from $\mathcal{C}$, i.e.

$$\boldsymbol{dl} = \begin{pmatrix} \vdots \\ dl_i \\ \vdots \end{pmatrix}, \quad i \in \mathcal{C};$$

- $d\boldsymbol{k}$ — the column of differentials of lengths for edges from $\overline{\mathcal{C}}$, i.e.

$$d\boldsymbol{k} = \begin{pmatrix} \vdots \\ dl_i \\ \vdots \end{pmatrix}, \quad i \in \overline{\mathcal{C}};$$

- $d\boldsymbol{\omega}$ — the column of differentials of curvatures around edges from $\mathcal{C}$;
- $d\boldsymbol{\psi}$ — the column of differentials of curvatures around edges from $\overline{\mathcal{C}}$.

**Lemma 2.** *Matrix $A$ (introduced in (11)), if written in a block form corresponding to the above mentioned partitions of sets of differentials:*

$$\begin{pmatrix} d\boldsymbol{\omega} \\ d\boldsymbol{\psi} \end{pmatrix} = A \begin{pmatrix} d\boldsymbol{l} \\ d\boldsymbol{k} \end{pmatrix}, \tag{21}$$

*has the following block structure:*

$$A = \begin{pmatrix} A|_{\mathcal{C}} & -(A|_{\mathcal{C}})\boldsymbol{a} \\ -\boldsymbol{a}^{\mathrm{T}}(A|_{\mathcal{C}}) & \boldsymbol{a}^{\mathrm{T}}(A|_{\mathcal{C}})\boldsymbol{a} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\boldsymbol{a}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} A|_{\mathcal{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & -\boldsymbol{a} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{22}$$

*where $\boldsymbol{a}$ is the matrix connecting $d\boldsymbol{k}$ and $d\boldsymbol{l}$ in the flat case:*

$$d\boldsymbol{l}|_{d\boldsymbol{\omega}=0} = \boldsymbol{a}\, d\boldsymbol{k} \tag{23}$$

*(recall that it follows from $d\boldsymbol{\omega} = 0$ that $d\boldsymbol{\psi} = 0$ as well); the superscript* T *means matrix transposing.*

**Proof.** Introduce (in analogy with the situation of adding a "new" edge in Section 3) the following column of differentials:

$$d\tilde{\boldsymbol{l}} = d\boldsymbol{l} - \boldsymbol{a}\, d\boldsymbol{k}. \tag{24}$$

Then if $d\tilde{\boldsymbol{l}}$ is zero, all curvatures vanish as well. This can be written as

$$\begin{pmatrix} d\boldsymbol{\omega} \\ d\boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} B & \mathbf{0} \\ C & \mathbf{0} \end{pmatrix} \begin{pmatrix} d\tilde{\boldsymbol{l}} \\ d\boldsymbol{k} \end{pmatrix}, \tag{25}$$

where $B$ and $C$ are some matrices, and it will be clear soon that $B = A|_{\mathcal{C}}$.

On the other hand, (24) can be rewritten as

$$\begin{pmatrix} d\tilde{\boldsymbol{l}} \\ d\boldsymbol{k} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\boldsymbol{a} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} d\boldsymbol{l} \\ d\boldsymbol{k} \end{pmatrix}. \tag{26}$$

Comparing (25) and (26) on the one hand, and (21) on the other, we see that

$$A = \begin{pmatrix} B & -B\boldsymbol{a} \\ C & -C\boldsymbol{a} \end{pmatrix}.$$

It is clear now that $B = A|_{\mathcal{C}}$, while the blocks in the second row of matrix $A$ are determined from the fact that it is symmetric (Theorem 2 and the Addition to it). The lemma is proved.

Now we are ready to investigate how $f = \det(A|_{\mathcal{C}})$ changes under replacing $\mathcal{C}$ with a similar set $\mathcal{C}'$. Such a $\mathcal{C}'$ can be obtained like this: choose subsets $\mathcal{A} \in \mathcal{C}$ and $\mathcal{B} \in \overline{\mathcal{C}}$ with the same number of elements, and move $\mathcal{A}$ from $\mathcal{C}$ to $\overline{\mathcal{C}}$, while $\mathcal{B}$ — vice versa. $\mathcal{C}$ is thus transformed into a set which we can take for $\mathcal{C}'$ ($\mathcal{A}$ and $\mathcal{B}$ must be chosen, however, in such a way that $\det(A|_{\mathcal{C}'})$ be not identically zero).

Denote also

$$f' = \det(A|_{\mathcal{C}'}) .$$

**Lemma 3.**

$$\frac{f'}{f} = \left(\det({}_{\mathcal{A}}|\boldsymbol{a}|_{\mathcal{B}})\right)^2 , \tag{27}$$

*where ${}_{\mathcal{A}}|\boldsymbol{a}|_{\mathcal{B}}$ means the square submatrix of $\boldsymbol{a}$ for which the rows corresponding to edges from $\mathcal{A}$ and the columns corresponding to edges from $\mathcal{B}$ are taken.*

**Proof.** One can see from the form (22) of matrix $A$ that $A|_{\mathcal{C}'}$ can be expressed through $A|_{\mathcal{C}}$. Namely,

$$A|_{\mathcal{C}'} = F^{\mathrm{T}} A|_{\mathcal{C}} F, \tag{28}$$

where the square matrix $F$ has the following block structure that corresponds to set partition $\mathcal{C} = \mathcal{C}\backslash\mathcal{A} \cup \mathcal{A}$ for the rows and $\mathcal{C}' = \mathcal{C}'\backslash\mathcal{B} \cup \mathcal{B}$ for the columns:

$$F = \begin{pmatrix} \mathbf{1} & * \\ \mathbf{0} & -{}_{\mathcal{A}}|\boldsymbol{a}|_{\mathcal{B}} \end{pmatrix} \tag{29}$$

(note that the *whole* matrix $F$ is thus of the same size as the upper left blocks e.g. in formula (22)). Here the asterisk means some submatrix that we are not interested in. By applying (28) and (29) the lemma is proved.

It follows from the definition (23) of matrix $\boldsymbol{a}$ connecting length differentials for zero curvatures that the determinant in the r.h.s. of (27) is nothing but

$$\det({}_{\mathcal{A}}|\boldsymbol{a}|_{\mathcal{B}}) = \pm \frac{\bigwedge\limits_{i\in\mathcal{A}} dl_i}{\bigwedge\limits_{i\in\mathcal{B}} dl_i} = \pm \frac{\bigwedge\limits_{i\in\overline{\mathcal{C}'}} dl_i}{\bigwedge\limits_{i\in\overline{\mathcal{C}}} dl_i},$$

where all $dl_i$ are taken as well for zero curvatures, i.e. they are forms on the space of *permitted length configurations*. They are nonzero due to Lemma 1. Hence, an important conclusion follows: *the form*

$$\frac{1}{\sqrt{|f|}} \bigwedge_{i\in\overline{\mathcal{C}}} dl_i, \quad \textit{or simply} \quad \frac{1}{\sqrt{|f|}} \bigwedge_{\overline{\mathcal{C}}} dl,$$

*does not change, to within its sign, with a different choice of $\mathcal{C}$.* This conclusion can be united with Theorem 3 in the following way.

**Theorem 4.** *The following differential form on the variety of permitted length configurations on the edges of complex (the degree of the form coincides with the dimension of* *variety) remains unchanged under* $2 \leftrightarrow 3$ *Pachner moves and under a different choice of subset* $\mathcal{C}$ *in the set of edges:*

$$\left| \frac{\prod\limits_{\text{over all edges}} l \bigwedge\limits_{\overline{\mathcal{C}}} dl}{\sqrt{|f|} \prod\limits_{\text{over all tetrahedra}} 6V} \right| . \tag{30}$$

Consider now a $1 \to 4$ Pachner move. This means that a tetrahedron $ABCD$ is replaced with four tetrahedra $ABCE$, $ABDE$, $ACDE$ and $BCDE$, where $E$ is a new vertex added to the complex. We will assume that edge $DE$ is added to the set $\mathcal{C}$, while edges $AE$, $BE$ and $CE$ are added to the set $\overline{\mathcal{C}}$. To trace the changes in the form (30), we have to note that $f$ is multiplied by $\partial\omega_{DE}/\partial l_{DE}$, and this partial derivative can be calculated again from formula (5) (although we are now in a somewhat different situation). As a result, (30) is multiplied by

$$\left| \frac{l_{AE}\, dl_{AE} \wedge l_{BE}\, dl_{BE} \wedge l_{CE}\, dl_{CE}}{6V_{ABCE}} \right| . \tag{31}$$

Assume that all edge lengths are temporarily fixed except $l_{AE}$, $l_{BE}$ and $l_{CE}$, and the complex is put into the 3-dimensional Euclidean space with a fixed coordinate system $Oxyz$ in such way that coordinates of all its vertices except $E$ are fixed. Then a simple trigonometry shows that (31) turns into

$$|dx_E \wedge dy_E \wedge dz_E| , \tag{32}$$

where, of course, $x_E$, $y_E$ and $z_E$ are Euclidean coordinates of point $E$.

## 5 The invariant $I$

Let us select three vertices among the vertices of the complex and denote them $A$, $B$ and $C$. Draw the axes $x$, $y$ and $z$ of a Euclidean system of coordinates in such way that $A$ be the origin of coordinates, $B$ lie on the $x$ axis and $C$ — in the plane $xAy$. When we vary the lengths of edges of the complex in a "permitted" way, the coordinates of vertices, namely $x_B$, $x_C$, $y_C$, $x_D$, $y_D$, $z_D, \ldots$ change, too.

**Lemma 4.** *The form*

$$\left| x_B^2\, dx_B \wedge dx_C \wedge y_C\, dy_C \wedge \bigwedge_{\textit{over remaining vertices}} dx \wedge dy \wedge dz \right| \tag{33}$$

*does not depend on the choice of $A$, $B$ and $C$.*

**Proof.** This simple fact can be proved in different ways. We prefer to link it to the ideas of paper [5].

Let us begin with the case where there is only one vertex $D$ in the complex besides $A$, $B$ and $C$. The already mentioned trigonometry (see (31) and (32)) shows that with fixed $A$, $B$ and $C$

$$|dx_D \wedge dy_D \wedge dz_D| = \frac{|l_{AD}\, dl_{AD} \wedge l_{BD}\, dl_{BD} \wedge l_{CD}\, dl_{CD}|}{6V_{ABCD}} .$$

Then, it is also easy to show that

$$|x_B^2\, dx_B \wedge dx_C \wedge y_C\, dy_C| = |l_{AB}\, dl_{AB} \wedge l_{AC}\, dl_{AC} \wedge l_{BC}\, dl_{BC}|. \tag{34}$$

Thus, the whole expression (33) is in this case equal to

$$\frac{|l_{AB}\, dl_{AB} \wedge \cdots \wedge l_{CD}\, dl_{CD}|}{6V_{ABCD}}, \tag{35}$$

with the exterior product in the enumerator taken over all edges of tetrahedron $ABCD$. It is clear that (35) does not change under all permutations of the set $\{A, B, C, D\}$.

Let now there be *five* vertices $A$, $B$, $C$, $D$ and $E$ in the complex. Then it is easy to show that (33) turns into

$$\frac{|l_{AB}\, dl_{AB} \wedge \cdots \wedge l_{CE}\, dl_{CE}|}{6V_{ABCD} \cdot 6V_{ABCE}}, \tag{36}$$

where the exterior product in the enumerator is taken over all edges entering in *at least one* of tetrahedra $ACBD$ and $ABCE$ (in other words, the edge $DE$ is absent from (36)). To conclude the proof for five vertices it is enough to show that (36) does not change under the permutation $C \leftrightarrow D$ (note that, again, $A$, $B$ and $C$ can be interchanged freely because of (34)).

Under $C \leftrightarrow D$, the factor $V_{ABCE}$ in the denominator of (36) is replaced with $V_{ABDE}$, and $l_{CE}\, dl_{CE}$ in the enumerator is replaced with $l_{DE}\, dl_{DE}$. Now it remains to apply the formula

$$\left|\frac{l_{CE}\, dl_{CE}}{V_{ABCE}}\right| = \left|\frac{l_{DE}\, dl_{DE}}{V_{ABDE}}\right|,$$

compare [5, formula (12)].

Finally, if there are more than five vertices, the proof of the lemma is obtained by obvious generalization of the above arguments. The lemma is proved.

Now let us note that the degree of the form (33) increases or decreases under moves $1 \leftrightarrow 4$ in the same way as the degree of the form (30), and they both do not change under $2 \leftrightarrow 3$ moves. Thus, the *difference* of those degrees is already a manifold invariant! Below, however, we will concentrate on another invariant which is defined only for those manifolds that satisfy the following assumption.

**Assumption.** Let our PL manifold be such that the degrees of forms (33) and (30) coincide and, moreover, they are proportional in each point of the "permitted" variety.

This Assumption seems to be satisfied for manifolds with finite fundamental groups, see examples in Section 6.

Adopting this Assumption, we divide (30) by (33). In general, we expect to get some function of the edge lengths in the complex. It turns out, somewhat surprisingly, that it actually *does not depend on those lengths* and is thus a constant depending on the manifold only!

To see this, let us return to our arguments about the $1 \to 4$ move from the end of Section 4. When we add the point $E$, the form (30) is multiplied by the form (31) or, which is the same, (32). Thus, the ratio (30)/(33), firstly, does not change and, secondly, is obviously independent of the position of point $E$ with respect to other vertices of the complex, that is of lengths $l_{AE}$, $l_{BE}$ and $l_{CE}$.

The point $E$ has however equal rights in this respect with the other vertices of the complex (any vertex can be eliminated, after some preparatory moves, by a $4 \to 1$ move. Thus, it can be regarded as added to some complex by the reverse $1 \to 4$ move). So, the ratio (30)/(33) (if it is well defined) does not depend at all on how we place the vertices in the Euclidean space.

**Definition.** *We define the invariant $I(M)$ for a given closed oriented PL manifold $M$ by the formula*

$$I(M) = \frac{(30)}{(33)},$$

*provided the right-hand side is well defined.*

## 6 Examples

In order to calculate the invariant $I$ for the sphere $S^3$, it is enough to use its decomposition in 2 tetrahedra. In such way we get a *pre-simplicial complex* (see e.g. [6]) rather than a simplicial complex, but our formulae remain valid for this case as well.

There will be 4 vertices, say $A, B, C$ and $D$; 6 edges which *all* enter the set $\overline{\mathcal{C}}$; 2 identical volumes $V_{ABCD}$; and the value $f$, due to the emptiness of the set $\mathcal{C}$, will be simply 1 (so, the rank of matrix $A = (\partial\omega_j/\partial l_k)$ will be zero). Thus, the formula (30) will yield in this case exactly the expression (35). This means that

$$I\left(S^3\right) = 1.$$

Consider now the projective space $RP^3$. For it, we take a triangulation that has, again, 4 vertices $A$, $B$, $C$ and $D$ but now 12 edges and 8 tetrahedra, see Figure 3. The edges will be distributed between sets $\mathcal{C}$ and $\overline{\mathcal{C}}$ as follows: $\mathcal{C} = \{b, c, d, f, g, h\}$; $\overline{\mathcal{C}} = \{b', c', d', f', g', h'\}$.

**Remark.** Figure 3 represents the "abstract triangulation" of $RP^3$ and does *not* depict the imbedding of the vertices in Euclidean space. Of course, such an imbedding cannot send, say, vertex $B$ in two different points.

Most partial derivatives of curvatures in edge lengths in Figure 3 are zero. We will explain this on the example of $\partial\omega_d/\partial l_f$. There are exactly two tetrahedra containing both edges $d$ and $f$. The dihedral angles at edge $d$ in those tetrahedra coincide in absolute value but have opposite signs, and this remains so even when we vary $l_f$ (in this case, it means that when taking the derivative $\partial\omega_d/\partial l_f$ we fix all lengths except $l_f$ so that $l_b = l_{b'}$, $l_c = l_{c'}$, $l_d = l_{d'}$, $l_g = l_{g'}$ and $l_h = l_{h'}$ but allow $l_f$ to vary in a neighborhood of $l_{f'}$).

Arguments of such sort show that the block $A|_{\mathcal{C}}$ of matrix $A = (\partial\omega_i/\partial l_j)$ contains only 6 nonzero entries, namely

$$\frac{\partial\omega_b}{\partial l_f}, \quad \frac{\partial\omega_f}{\partial l_b}, \quad \frac{\partial\omega_c}{\partial l_g}, \quad \frac{\partial\omega_g}{\partial l_c}, \quad \frac{\partial\omega_d}{\partial l_h} \quad \text{and} \quad \frac{\partial\omega_h}{\partial l_d}, \tag{37}$$

so that $\det A|_{\mathcal{C}}$ coincides in absolute value with the product of them all.

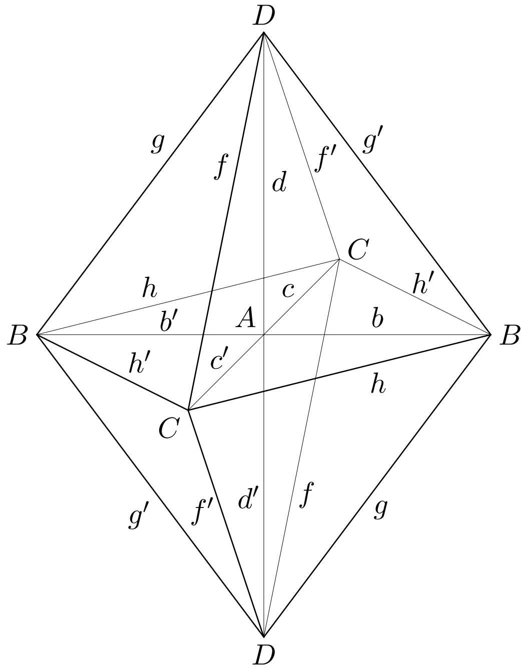


**Figure 3.** Triangulation for $RP^3$

In order to calculate, say, $\partial\omega_b/\partial l_f$ we note that there exist exactly two tetrahedra containing both edges $b$ and $f$, and one can see from Figure 3 that in these tetrahedra the derivatives of dihedral angles at edge $b$ in $l_f$ have the *same* sign. They are calculated by formulae of type (3), so that we get

$$\left|\frac{\partial\omega_b}{\partial l_f}\right| = 2\cdot\left|\frac{l_b\, l_f}{6\, V_{ABCD}}\right|. \tag{38}$$

The rest of derivatives (37) are calculated in a similar way, and due to the multiplication by 2 in (38) and other such formulas the form (30) turns out to be 1/8 of the similar form for $S^3$, which means that

$$I\left(RP^3\right) = \frac{1}{8}.$$

## 7 Discussion

In this Letter we have only got some first results showing that it is possible to construct, on the base of such quantities as edge lengths, dihedral angles and volumes, an invariant which can be calculated with no complications at least for the sphere $S^3$ and projective space $RP^3$. Actually, some calculations have been done also for lens spaces $L(p,q)$, and they suggest that

$$I(L(p,q)) = \frac{1}{p^3}. \tag{39}$$

Thus, this version of invariant seems, somewhat regretfully, not to depend on $q$.

It must not be forgotten however that the presented version of invariant is only the simplest one (and actually another invariant has been already mentioned in Section 5, before the Assumption). An interesting possibility is to map in $R^3$ not a simplicial complex itself but its *universal covering*, in case of a nontrivial fundamental group $\pi_1$.

This idea will most likely be combined with the form (33) being replaced with some other "standard" differential form, of the same degree as (30). So, we will be led probably to richer invariant structures than just a number like (39).

The work [5] suggests that invariants of the same type as in this Letter can be constructed for higher-dimensional manifolds as well. This may be combined with other ways of generalization such as the use of noncommutative "lengths".

Finally, the fact that the tetrahedron volumes enter in formula (30) raised in the power $(-1/2)$ shows that our formulae are akin to the quasiclassical formulae suggested by Ponzano and Regge [2] and proved in a recent work by Justin Roberts [3] (where very interesting mathematical facts are presented related to $6j$-symbols and Euclidean geometry).

**Acknowledgements**

I am grateful to many people for scientific discussions, explaining me topological and other subtleties and simply for encouraging letters. Among them are: Rinat Kashaev, Greg Kuperberg, Marco Mackaay, Sergei Matveev, Justin Roberts, Satoru Saito, and Jim Stasheff. I also acknowledge the partial financial support from Russian Foundation for Basic Research under grant no. 01-01-00059.

## References

[1] Regge T, General Relativity without Coordinates, *Nuovo Cim.* **A19** (1961), 558–571.

[2] Ponzano G. and Regge T., Semiclassical Limit of Racah Coefficients, in Spectroscopic and Group Theoretical Methods in Physics, edited by F Bloch, North Holland - Amsterdam, 1968.

[3] Roberts J D, Classical $6j$-Symbols and the Tetrahedron, *Geometry and Topology* **3** (1999), 21–66.

[4] Korepanov I G, A Formula with Volumes of Five Tetrahedra and Discrete Curvature, preprint nlin.SI/0003001.

[5] Korepanov I G, Multidimensional Analogues of the Geometric $s \leftrightarrow t$ Duality, *Teor. Mat. Fiz.* **124**, No.2 (2000), 169–176 [English translation *Theor. Math. Phys.* **124** No.1 (2000), 999–1005].

[6] Hilton P J and Wylie S, Homology Theory. An Introduction to Algebraic Topology, Cambridge University Press, 1960.