An application of classical analysis to intertemporal choice

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What is this talk is about?

- The talk is about utilitarianism.
- Utilitarianism is about utility (welfare/happiness/well-being).
- Classic utilitarianism concerns finite societies.

  For a society with N members, let $u = (u_1, u_2, \ldots, u_N)$ and $v = (v_1, v_2, \ldots, v_N)$ be the utility profiles of two social states.

Classic utilitarianism

State $u = (u_1, u_2, \ldots, u_N)$ is at least good as state $v = (v_1, v_2, \ldots, v_N)$ if and only if $\sum_{n=1}^{N} u_n \geq \sum_{n=1}^{N} v_n$.

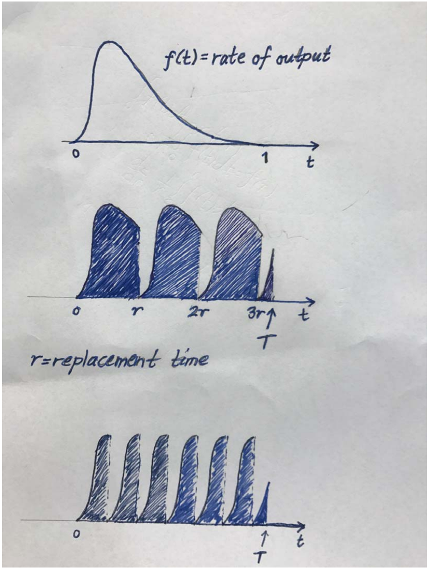
Problem

What if society has infinitely many bearers of utility? For example, is $u = (1, 1, 1, ...)$ better than $v = (0, 1, 1, 1, ...)$? If so, why?

Motivation: Infinite-horizon intertemporal choice

- Moral philosophy (Sidgwick, 1907; Segerberg, 1976; Vallentyne, 1994)
- Economic growth (Ramsey, 1928; von Weizsäcker, 1965; Gale, 1967)
- Welfare economics (Pigou, 1920; Koopmans, 1960; Stern, 2006)
- Game theory (Rubinstein, 1979; Aumann and Shapley, 1992)
- Markov decision processes (Blackwell, 1962; Veinott, 1966)

More motivation: A problem in maintenance engineering



More motivation: a problem in maintenance engineering

- *Discounted utilitarianism*: the maximand is

$$\int_0^\infty \delta^t u(t)dt,$$

  where $u\colon [0,\infty) \to \mathbb{R}$ is the "reward" or "utility" function and where $\delta \in (0,1)$ is the "discount factor".

- In the optimal replacement problem the reward function is periodic with period r:

  $u(t) = f(t)$ if $0 \leq t < r$,

  $u(t) = f(t - jr)$ if $jr \leq t < (j+1)r, j = 1, 2, 3, ...$.

More motivation: a problem in maintenance engineering

- The reward function is periodic with period r:
  $u(t) = f(t)$ if $0 \leq t < r$,
  $u(t) = f(t - jr)$ if $jr \leq t < (j+1)r, j = 1, 2, 3, \ldots.$
- Thus we want to find r to maximize (with δ fixed)

$$\begin{aligned} V(r) \equiv \int_0^\infty \delta^t u(t)dt &= \sum_{j=0}^{\infty} \int_{jr}^{(j+1)r} \delta^t u(t)dt \\ &= \sum_{j=0}^{\infty} \int_0^r \delta^{t+jr} f(t)dt \\ &= \sum_{j=0}^{\infty} \delta^{jr} \int_0^r \delta^t f(t)dt \\ &= \frac{1}{1-\delta^r} \int_0^r \delta^t f(t)dt. \end{aligned}$$

More motivation: a problem in maintenance engineering

We use Calculus to maximize

$$V(r) = \frac{1}{1-\delta^r} \int_0^r \delta^t f(t) dt.$$

By the product rule and the Fundamental Theorem, we have

$$\frac{d}{dr}\left(\frac{1}{1-\delta^r}\int_0^r \delta^t f(t)dt\right) = \frac{\log(\delta)\delta^r}{(1-\delta^r)^2}\int_0^r \delta^t f(t)dt + \frac{1}{1-\delta^r}\delta^r f(r)$$

and

$$V''(r) < 0.$$

So the optimal r solves the equation

$$\frac{\log(\delta)\delta^r}{(1-\delta^r)^2}\int_0^r \delta^t f(t)dt + \frac{1}{1-\delta^r}\delta^r f(r) = 0. \tag{1}$$

More motivation: a problem in maintenance engineering

Condition for $V'(r) = 0$:

$$\frac{\log(\delta)\delta^r}{(1-\delta^r)^2} \int_0^r \delta^t f(t)dt + \frac{1}{1-\delta^r}\delta^r f(r) = 0.$$

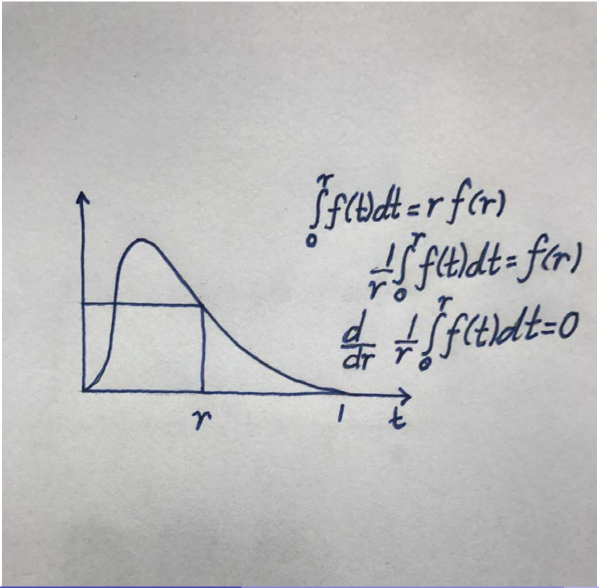
Multiply by $(1-\delta^r)/\delta^r$ to get

$$\frac{\log(\delta)}{1-\delta^r} \int_0^r \delta^t f(t)dt + f(r) = 0. \tag{2}$$

As $\delta \uparrow 1$, we have $\frac{\log(\delta)}{1-\delta^r} \to -1/r$ and $\int_0^r \delta^t f(t)dt \to \int_0^r f(t)dt$. So in the limit, (2) says that

$$\frac{1}{r} \int_0^r f(t)dt = f(r). \tag{3}$$

More motivation: A problem in maintenance engineering



Outline

- The standard model of intertemporal choice.
- The axiomatic approach to the intertemporal choice problem.
- The single-agent version of the intertemporal choice problem.
- Some philosophical and sociological aspects of the intergenerational version of the problem.

The standard model of intertemporal choice

- $\mathscr{U}$ is the set of all sequences $u = (u_1, u_2, u_3, \dots)$, each $u_t \in \mathbb{R}$.
- We interpret $u = (u_1, u_2, u_3, \dots) \in \mathscr{U}$ as a stream of payoffs, or utilities, that an agent receives in successive non-overlapping time periods, where u_t is the payoff (or utility) in period t.
- The agent can, for example, be a player in an infinite-horizon game or be a social planner for an infinite-horizon society.
- The agent's preferences are defined by a binary relation $\succsim$ on $\mathscr{U}$:
  $u \succsim v$ means that u is at least as good as v.
  $u \succ v$ means that u is better than v ($u \succsim v$ but not $v \succsim u$).
  $u \sim v$ means that u and v are equally good ($u \succsim v$ and $v \succsim u$).
- $\succsim$ is reflexive ($u \sim u$ for all $u \in \mathscr{U}$) and transitive (if $u \succsim v$ and $v \succsim w$, then $u \succsim w$).

Minimum requirements

Axiom **A1** (Monotonicity/Pareto): for all $u, v \in \mathscr{U}$, if $u_t \geq v_t$ for all t and $u_t > v_t$ for at least one t, then $u \succ v$.

So, for example, $u = (1, 1, 1, ...)$ is better than $v = (0, 1, 1, 1, ...)$ by **A1**.

Axiom **A2** (Translation scale invariance/Interpersonal comparability of utility): for all $u, v, \alpha \in \mathscr{U}$, if $u \succsim v$, then $u + \alpha \succsim v + \alpha$.

Axiom **A3** (Time Neutrality/Anonymity) says that players are "infinitely patient"/generations are treated equally: for all $u, v \in \mathscr{U}$, if u can be obtained from v by interchanging two of v's entries, then $u \sim v$.

So, for example, $(1, 0, 0, 0, ...) \sim (0, 1, 0, 0, 0, ...)$ by **A3**.

A2: Translation scale invariance

A2: for all $u, v, \alpha \in \mathscr{U}$, if $u \succsim v$, then $u + \alpha \succsim v + \alpha$.
If $\succsim$ satisfies **A2**, then

$$u \succsim v \iff u - v \succsim (0, 0, 0, \ldots):$$

$$\text{for the implication } u \succsim v \Longrightarrow u - v \succsim (0, 0, 0, \ldots), \text{ take } \alpha = -v,$$
$$\text{for the implication } u - v \succsim (0, 0, 0, \ldots) \Longrightarrow u \succsim v, \text{ take } \alpha = v.$$

If **A3** holds we have

$$(-1/2, 1/2, 0, 0, \ldots) \sim (1/2, -1/2, 0, 0, 0, \ldots). \tag{4}$$

By **A2**, adding $\alpha = (1/2, -1/2, 0, 0, 0, \ldots)$ to both sides of (4) gives

$$(0, 0, 0, \ldots) \sim (1, -1, 0, 0, 0, \ldots) \tag{5}$$

and adding $\alpha' = (1, 1, 0, 0, 0, \ldots)$ to both sides of (5) gives

$$(1, 1, 0, 0, 0, \ldots) \sim (2, 0, 0, 0, \ldots).$$

A1-A3

Assume that **A1-A3** hold. Then we have (d'Aspremont and Gevers, 1977) that for all $u, v \in \mathscr{U}$ with at most a finite number of non-zero entries,

$$u \succsim v \iff \sum_{t=1}^{\infty} u_t \geq \sum_{t=1}^{\infty} v_t.$$

More generally, if $u_t - v_t = 0$ for all but finitely many values of t, then

$$u \succsim v \iff \sum_{t=1}^{\infty} (u_t - v_t) \geq 0.$$

But we don't get much more.

Some restrictions

Regarding preference relations satisfying **A1** and **A3**:

- They cannot be defined by an objective function $f\colon \mathscr{U} \to \mathbb{R}$.
  That is, for any $f\colon \mathscr{U} \to \mathbb{R}$, the relation $u \succsim v \Leftrightarrow f(u) \geq f(v)$ does not satisfy both **A1** and **A3** (Diamond, 1965; Basu and Mitra, 2003)
- Complete preferences do exist (Svensson, 1980).
- But they cannot be defined explicitly. That is, there are no constructive existence proofs (Zame, 2007; Lauwers, 2010).

We therefore settle for incomplete preferences.

Examples: Long-run averages

We might consider the *long-run average criterion*: for $u, v \in \mathscr{U}$, if

$$\bar{u} \equiv \lim_{n\to\infty} \sum_{t=1}^{n} u_t/n$$

and $\bar{v} \equiv \lim_{n\to\infty} \sum_{t=1}^{n} v_t/n$ are well defined and finite, then

$$u \succsim v \iff \bar{u} \geq \bar{v}.$$

But then $(1, 1, 1, ...)$ and $(0, 1, 1, 1, ...)$ become equally good. So **A1** fails.

Examples: overtaking

Alternatively we might consider the *overtaking criterion* (Ramsey, 1928; von Weizsäcker, 1965; Gale, 1967; Brock, 1970):

$$u \succsim v \iff \exists T_0 \in [0, \infty) \text{ such that } \sum_{t=1}^{T} (u_t - v_t) \geq 0 \text{ for all } T \geq T_0.$$

A slightly less selective (less incomplete) version is given by

$$u \succsim v \iff \liminf_{T \to \infty} \sum_{t=1}^{T} (u_t - v_t) \geq 0.$$

Still not good enough for many practical purposes:

- If $u = (1, 0, 1, 0, \dots)$ and $c \in (0, 1)$, then u and (c, u) can not be compared: $\sum_{t=1}^{T} (u_t - v_t)$ oscillates between $1/2$ and $-1/2$.
- There are Markov decision processes with finite state and action spaces where no overtaking optimal policy exists (Brown, 1965; Nowak and Vega-Amaya, 1992).

The Compensation Principle

Imagine an infinitely patient player in an infinite-horizon game or discrete time Markov decision process.

- For $u \in \mathscr{U}$ and $c \in \mathbb{R}$, let $(c, u) \equiv (c, u_1, u_2, u_3, \dots)$ be the postponement of u with compensation c.
- Which compensation should the agent accept?

  That is, for which value of c should $(c, u) \sim u$ hold?

  More generally, which values of c should ensure $(c, u) \succsim u$?

The Compensation Principle

- If $u = (1, 0, 0, \dots)$, then $(0, u) = (0, 1, 0, 0, \dots) \sim u$ by **A3**.

  So no compensation required.

  Same conclusion if u has a finite number of nonzero entries.
- If $u \in \mathscr{U}$ is summable (i.e., $\sum_{t=1}^{\infty} u_t$ is convergent), then $(0, u) \sim u$ by the principle that streams with the same finite sums are equivalent.
- If $u = (1, 1, 1, \dots)$, then $(0, u) = (0, 1, 1, 1, ...)$ is worse than u by **A1**, and $(c, u) \sim u$ can only hold if $c = 1$.
- If $u = (0, 1, 1, 1, \dots)$, then $(c, u) \sim u$ if $c = 1$ by **A3**.
- If $u = (1, 0, 1, 0, \dots)$, then **A1**-**A3** provide no guidance.

  But $(c, c, u) \sim u$ can only hold if $c = 1/2$.
- In each case above, $c = \bar{u}$.

CP: The Compensation Principle (Jonsson and Voorneveld, 2018)

For all $u \in \mathscr{U}$, if $\bar{u}$ is defined and finite, then $(\bar{u}, u) \sim u$.

A1+A2+CP

For $u, v \in \mathscr{U}$, let $s = (s_1, s_2, s_3, \ldots)$ be the sequence of partial sums of $u - v$. That is,

$$s_n \equiv \sum_{t=1}^{n} (u_t - v_t), n \geq 1.$$

Observation:

$$u - v = s - (0, s).$$

Indeed, the first component of $s - (0, s)$ is

$$s_1 = u_1 - v_1.$$

The second component of $s - (0, s)$ is

$$s_2 - s_1 = u_2 - v_2.$$

For $j \geq 2$, the j:th component of $s - (0, s)$ is

$$s_j - s_{j-1} = u_j - v_j.$$

A1+A2+CP

We have

$$u - v = s - (0, s). \tag{6}$$

By **A2** (Translation scale invariance/Interpersonal comparability),

$$u \succsim v \iff u - v \succsim (0, 0, 0, \ldots).$$

By (6),

$$u \succsim v \iff s - (0, s) \succsim (0, 0, 0, \ldots).$$

By **A2**,

$$s - (0, s) \succsim (0, 0, 0, \ldots) \iff s \succsim (0, s).$$

Thus

$$u \succsim v \iff s \succsim (0, s).$$

A1+A2+CP

By **A2** (Translation scale invariance),

$$u \succsim v \iff s \succsim (0, s). \tag{7}$$

Suppose that $\bar{s}$ is well defined and finite. Then

$$s \sim (\bar{s}, s) \tag{8}$$

by **CP**. By (8) and **A1** (Monotonicity),

$$s \succsim (0, s) \iff \bar{s} \geq 0. \tag{9}$$

Combining (7) and (9) gives

$$u \succsim v \iff \bar{s} \geq 0.$$

That is,

$$u \succsim v \iff \lim_{n\to\infty} \frac{1}{n} \sum_{T=1}^{n} \sum_{t=1}^{T} (u_t - v_t) \geq 0.$$

Average overtaking

Axioms **A1**, **A2**, and **CP** generate the preference relation

$$u \succsim v \iff \lim_{n\to\infty} \frac{1}{n} \sum_{T=1}^{n} \sum_{t=1}^{T} (u_t - v_t) \geq 0.$$

The *average overtaking criterion*

$$u \succsim v \iff \liminf_{n\to\infty} \frac{1}{n} \sum_{T=1}^{n} \sum_{t=1}^{T} (u_t - v_t) \geq 0$$

was introduced by Veinott (1966) for Markov decision processes (MDPs).
It was characterized by Jonsson (2017).
Good enough for MDPs with finite state and action spaces (Denardo and Miller, 1968).

Summability

Let $a = (a_1, a_2, a_3, ...) \in \mathscr{U}$.
If the limit

$$\sigma(a) \equiv \lim_{T\to\infty} \sum_{t=1}^{T} a_t$$

exists and is finite, then $\sum_{t=1}^{\infty} a_t$ is said to be convergent or summable. If not, the series is usually said to be *divergent*.
If the limit

$$\sigma_C(a) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{T=1}^{n} \sum_{t=1}^{T} a_t$$

exists and is finite, then $\sum_{t=1}^{\infty} a_t$ is said to be *Cesàro-summable* to $\sigma_C(a)$.
If the limit

$$\sigma_A(a) \equiv \lim_{\delta\to 1^-} \sum_{t=1}^{\infty} \delta^t a_t$$

exists and is finite, then $\sum_{t=1}^{\infty} a_t$ is said to be *Abel-summable* to $\sigma_A(a)$.

Another criterion

The "limit-discounted" criterion (Jonsson and Voorneveld, 2018):

$$u \succsim v \iff \liminf_{\delta \to 1^-} \sum_{t=1}^{\infty} \delta^t (u_t - v_t) \geq 0.$$

- Closely related to Blackwell's (1962) concept of 1-optimality.
- Related to Veinott's criterion:

  A Cesàro-summable series is Abel-summable to the same sum (Frobenius, 1880).

Abel - and Cesàro-limits

- In general (for any $u, v \in \mathscr{U}$),

$$\liminf_{\delta \to 1^-} \sum_{t=1}^{\infty} \delta^t (u_t - v_t) \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{T=1}^{n} \sum_{t=1}^{T} (u_t - v_t). \tag{10}$$

- If the partial sums $s_n = \sum_{t=1}^{n} (u_t - v_t), n \geq 1$, are bounded below or above and $\lim_{\delta \to 1^-} \sum_{t=1}^{\infty} \delta^t (u_t - v_t)$ exists, then the inequality in (10) is an equality (Hardy and Littlewood, 1914).
- If the partial sums $s_n = \sum_{t=1}^{n} (u_t - v_t), n \geq 1$, are bounded below or above, then the inequality in (10) is an equality (1957).
- We may have strict inequality in (10) even with bounded partial sums (Liggett and Lippman, 1969; Bishop et al, 2014).

Theorem

Theorem

Let $\mathscr{U}_0$ be the set of summable or eventually periodic $u \in \mathscr{U}$, and let $\succsim$ and $\succsim'$ be any two preference relations satisfying **A1**, **A2**, and **CP**. Then $\succsim$ and $\succsim'$ rank all pairs $u, v \in \mathscr{U}_0$, and $u \succsim v \iff u \succsim' v$.

On **A3**

A1 (Pareto): If $u_t \geq v_t$ for all t and $u_t > v_t$ for some t, then $u \succ v$.
A2 (Interpersonal comparability of utility): If $u \succsim v$, then $u + \alpha \succsim v + \alpha$.
A3 (Anonymity): If u can be obtained from v by swapping two entries, then $u \sim v$.

Since preferences are assumed transitive, **A3** is equivalent to the condition that $u \sim v$ holds whenever u is a finite permutation of v.

A permutation is a one-to-one onto map $\pi : \mathbb{N} \equiv \{1, 2, 3, ...\} \to \mathbb{N}$.

A permutation is finite if it agrees with the identity off a finite set $F \subset \mathbb{N}$.

A permutation is infinite if it is not finite.

Call $v \in \mathscr{U}$ a permutation of $u \in \mathscr{U}$ if $v = \pi(u) \equiv (u_{\pi(1)}, u_{\pi(2)}, u_{\pi(3)}, ...)$.

On permutations and equity

"Extended" anonymity (1995)

For all $u \in \mathscr{U}$, $u \sim \pi(u)$ for all $\pi \in \mathscr{Q}$, where $\mathscr{Q}$ is a set of permutations where the finite permutations are strictly contained.

"Strong" anonymity (2012)

For all $u \in \mathscr{U}$, $u \sim \pi(u)$ for *all* permutations.

Sounds good. But,

$$u = (1, 0, 1, 0, 1, 0, \dots)$$
$$v = (0, 0, 1, 0, 1, 0, \dots)$$

are permutations of each other. Worse yet, the same is true of

$$u = (\overbrace{1, 1, 1, \dots, 1}^{\text{many ones}}, 0, \overbrace{1, 1, 1, \dots, 1}^{\text{many ones}}, 0, \overbrace{1, 1, 1, \dots, 1}^{\text{many ones}}, 0, \dots)$$
$$v = (1, \underbrace{0, 0, 0, \dots, 0}_{\text{many zeros}}, 1, \underbrace{0, 0, 0, \dots, 0}_{\text{many zeros}}, 1, \underbrace{0, 0, 0, \dots, 0}_{\text{many zeros}}, \dots)$$

On permutations and equity

Say we have two random samples: $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$.

u_1	u_2	u_3	...	u_n
v_1	v_2	v_3	...	v_n

- If we, for example, are comparing men and women, we use "completely randomized design"/"Två stickprov".

  Then it does not matter if we can interchange e.g. u_1 and u_2.
- If we, for example, are comparing blood pressure before and after treatment, we use "paired comparison design"/"Stickprov i par".

  Then it *does* matter if we can interchange u_1 and u_2 (unless we also interchange v_1 and v_2).

The prevalence of extended anonymity

- It is hard to explain why extended anonymity is wrong.
- Extended anonymity appears to be related to justice.
- Extended anonymity produces impossibility results.
- There is a widely discussed example that appears to justify extended anonymity, namely:

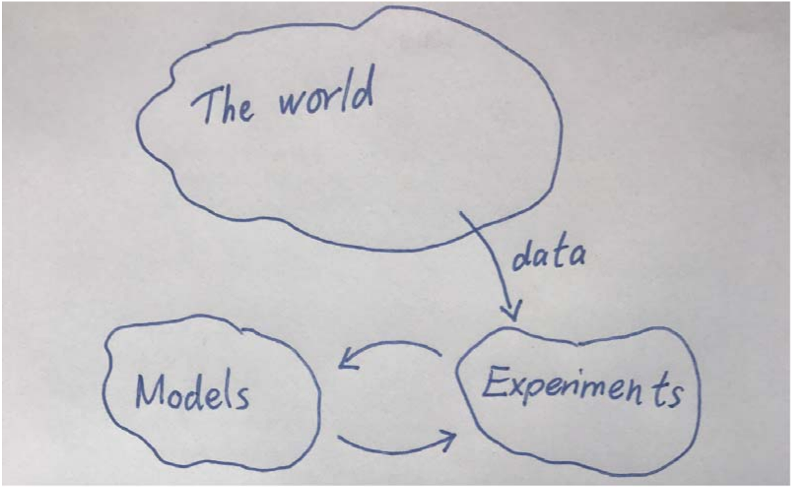
$$u = (1, 0, 1, 0, \dots)$$
$$v = (0, 1, 0, 1, \dots)$$

From the literature:

"$v = (0, 1, 0, ...)$ is the postponement of $u = (1, 0, 1, ...)$" (2005, p. 1115).

"...since utilitarianism judges (0,1) and (1,0) equally good, one might expect that this indifference extends to infinite repetitions" (1997, p. 225).

Reflective equilibrium and theory justification



Reflective equilibrium and theory justification

- Seeking reflective equilibrium means advancing arguments that bring out relative strengths and weaknesses of alternative principles and competing moral judgements (Rawls, 1971). But how do we make plausible that we are approaching reflective equilibrium?
- Daniels (1979) suggests that the arguments that bring out the relative strengths and weaknesses of alternative principles be interpreted as inferences from some set of relevant "background theories". He argues that in (wide) reflective equilibrium, we should not only have a fit between principles and seemingly appealing intuitions. The background theories...

  "...should show that our principles are more acceptable than alternative principles on grounds that are to some degree independent of their match with moral judgments".

A4

A1: If $u_t \geq v_t$ for all t and $u_t > v_t$ for some t, then $u \succ v$.
A2: If $u \succsim v$, then $u + \alpha \succsim v + \alpha$ for every $\alpha \in \mathscr{U}$.
A3: If u can be obtained from v by swapping two entries, then $u \sim v$.

A1-**A3** do not reflect that the utility bearers follow each other in time.

The following condition (cf. Koopmans, 1960) does:

A4 (Stationarity): for $u, v \in \mathscr{U}$ and $c \in \mathbb{R}$, if $u \succsim v$, then $(c, u) \succsim (c, v)$.

Theorem

Let $\succsim$ satisfy **A1** and **A4**.
Then $u = (1, 0, 1, 0, ...)$ and $(0, u)$ are not equivalent (Asheim et al, 2010).
Proof: If $u \sim (0, u)$, then $(0, u) \sim (0, 0, u)$ by **A4**. So $u \sim (0, 0, u)$ by transitivity: a violation of **A1**.

Theorem

Let $\succsim$ satisfy **A1**-**A4**. If $u \in \mathscr{U}$ is eventually periodic, then $(c, u) \sim u$ can only hold if $c = \bar{u}$.

So **CP** is supported independently.

Main references

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