Meteors and Celestial Dynamics

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Outline

1 Introduction
- The scenario
- The modeling

2 Theory
- Hamiltonian mechanics
- Hamiltonian splitting
- Deterministic chaos
- Some of the statistics

3 Software and application
- Software design
- 21P/Giacobini-Zinner

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Comets

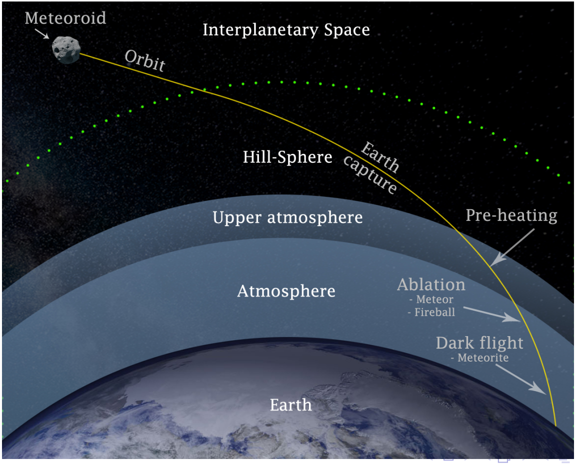
Credits: NASA

Formation of meteoroid streams

URL:

https://www.youtube.com/watch?v=KsLGKgdVBHQ&feature=youtu.be

Name that space rock



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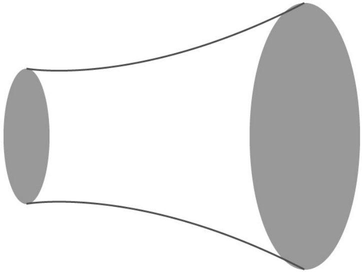
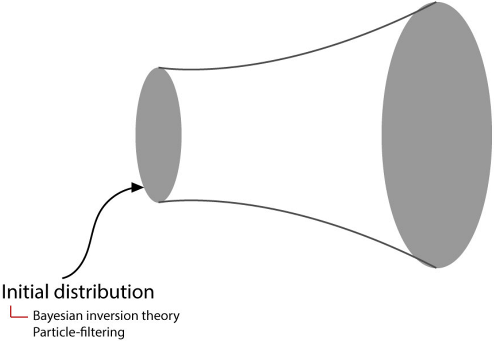
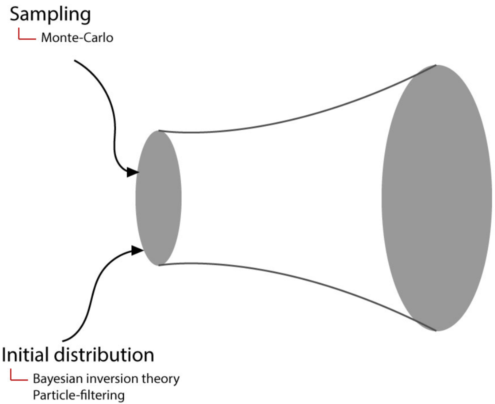
The setup



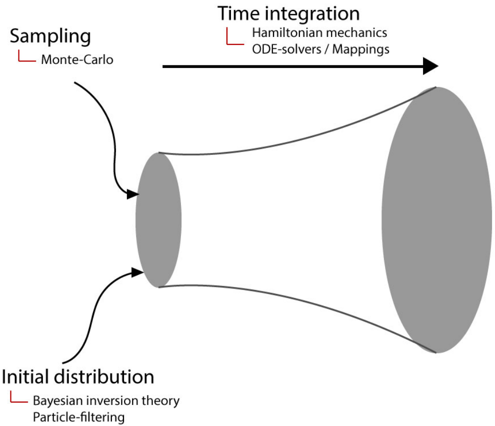
The setup


Initial distribution
Bayesian inversion theory
Particle-filtering

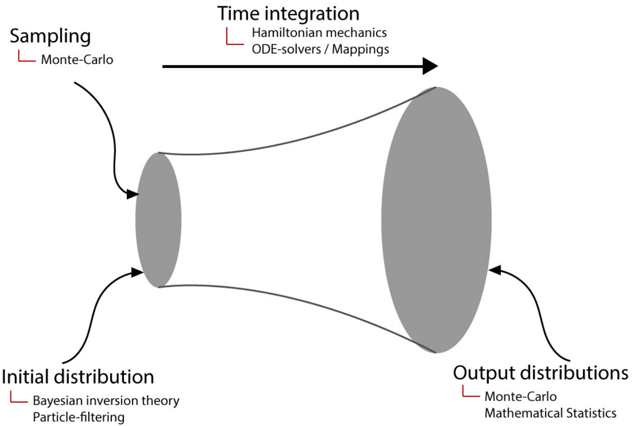
The setup


Sampling
Monte-Carlo
Initial distribution
Bayesian inversion theory
Particle-filtering

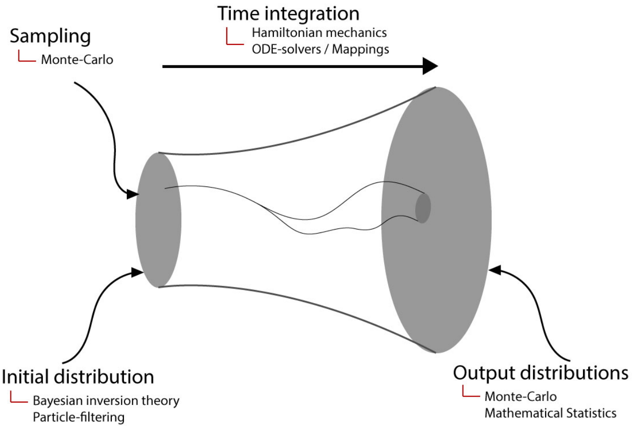
The setup


Time integration
Hamiltonian mechanics
ODE-solvers / Mappings
Sampling
Monte-Carlo
Initial distribution
Bayesian inversion theory
Particle-filtering

The setup


Time integration
Hamiltonian mechanics
ODE-solvers / Mappings
Sampling
Monte-Carlo
Initial distribution
Bayesian inversion theory
Particle-filtering
Output distributions
Monte-Carlo
Mathematical Statistics

The setup


Time integration
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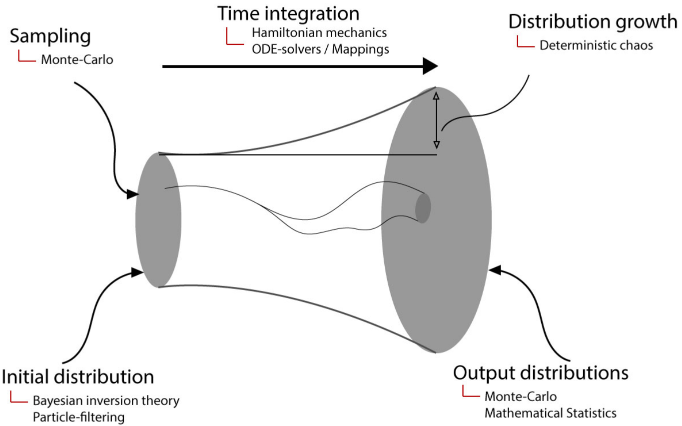
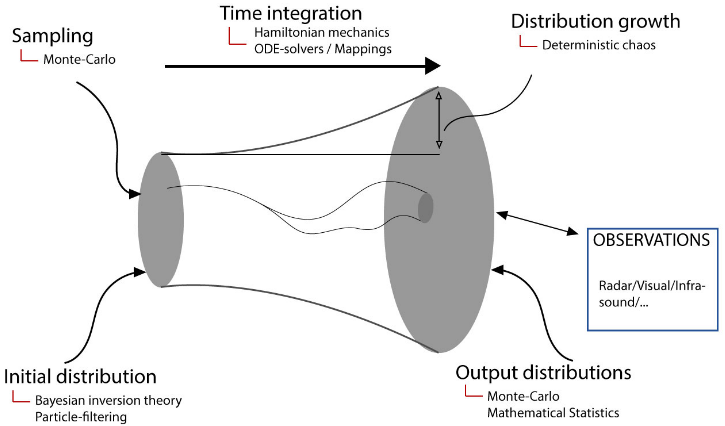
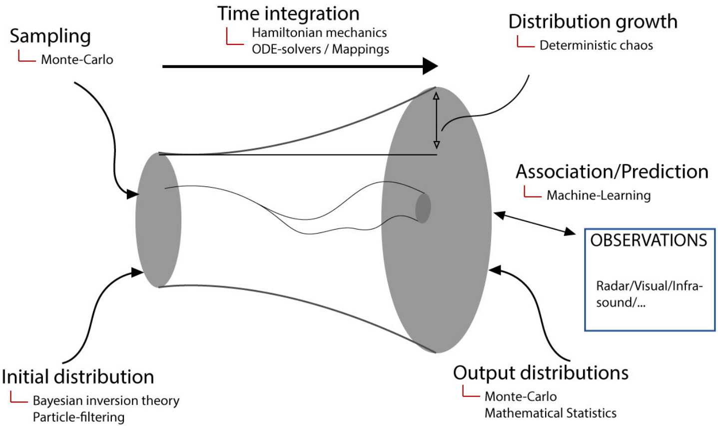

Sampling
Monte-Carlo
Time integration
Hamiltonian mechanics
ODE-solvers / Mappings
Distribution growth
Deterministic chaos
Initial distribution
Bayesian inversion theory
Particle-filtering
Output distributions
Monte-Carlo
Mathematical Statistics

The setup


Sampling
Monte-Carlo
Time integration
Hamiltonian mechanics
ODE-solvers / Mappings
Distribution growth
Deterministic chaos
OBSERVATIONS
Radar/Visual/Infra-
sound/...
Initial distribution
Bayesian inversion theory
Particle-filtering
Output distributions
Monte-Carlo
Mathematical Statistics

The setup


Sampling
Monte-Carlo
Time integration
Hamiltonian mechanics
ODE-solvers / Mappings
Distribution growth
Deterministic chaos
Association/Prediction
Machine-Learning
OBSERVATIONS
Radar/Visual/Infra-sound/...
Initial distribution
Bayesian inversion theory
Particle-filtering
Output distributions
Monte-Carlo
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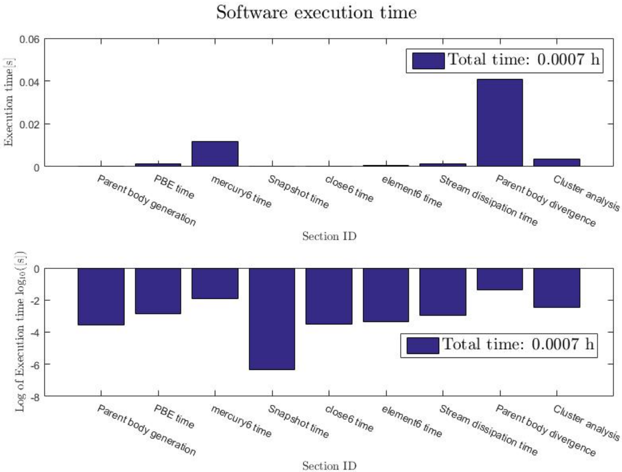
What to focus on!?



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Non-Hamiltonian perturbations

Gravity
- Newtonian (Hamiltonian)
- General relativity

Electromagnetic
- Photo/Plasma Poynting Robertson effect
- Yarkovsky effect
- YORP effect
- Radiation pressure

Differential equation flow

Let us assume a set of time dependant variables $\mathbf{x}(t)$ in a phase space M

$$\mathbf{x}(t) = \Psi_t \mathbf{x}(0), \tag{1}$$
$$\Psi_t : M \mapsto M. \tag{2}$$

The flow Ψ_t is not always known. Thus we try to find maps, Φ, to approximate this flow, e.g. with discrete steps

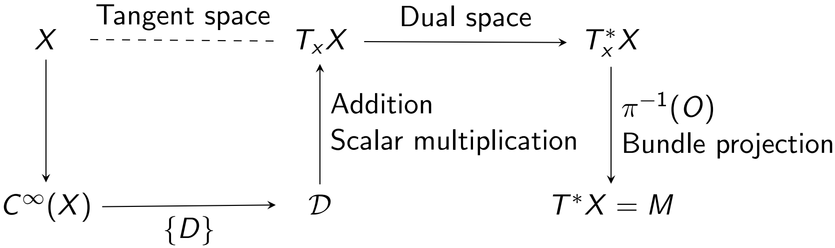
$$\Phi_h : (\mathbf{x}_n) \mapsto (\mathbf{x}_{n+1}) \tag{3}$$

Phase space?

A phase space M is constructed trough *Generalized coordinates* $q \in X$ and the momentum $p \in T_q^* X$

Phase space

Here X is the configuration space with base field F (in our case $\mathbb{R}$)



Informal: $T_x X$ all possible "directions" which one can tangentially pass through x

Differential forms

Differential forms are a way to describe multi-variable calculus independent of coordinates.

Differential forms

The tautological one-form is given by (remember tensors in GR)

$$\theta = \sum_i p_i \mathrm{d}q^i \tag{4}$$

Taking the exterior derivative of θ gives symplectic two-form (remember bi-vectors in geometric algebra)

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i \tag{5}$$

Hamiltonian form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}) \tag{6}$$

is on *Hamiltonian form* if $\mathbf{x} = \begin{pmatrix} q_1 & \dots & q_N & p_1 & \dots & p_N \end{pmatrix}^T$, and there exists a function $H(\mathbf{x}, t)$ such that

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \; \forall \; i \in 1, \dots, N, \tag{7}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \; \forall \; i \in 1, \dots, N. \tag{8}$$

Hamiltonian form

But it is deeper than than: any real function T^*X can be interpreted to be a Hamiltonian

Note for all you Lagrangian's out there:

$$L : TM \mapsto F \tag{9}$$

$$H : T^*M \mapsto F \tag{10}$$

and the Legendre transform $L \mapsto H$

Poisson brackets

Consider a function f and g on M, we can define the Poisson brackets as

$$\{f(\mathbf{x}), g(\mathbf{x})\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \tag{11}$$

Poisson brackets:
- Form a Lie algebra
- Are linear and anti-commuting in their arguments

Phase space paths

Full time derivative of a phase space path (governed by H) can be expressed by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \{\mathbf{x}, H\} + \frac{\partial \mathbf{x}}{\partial t}. \tag{12}$$

Propagator

Let us assume $t_0 = 0$. We can show that $\frac{\partial \mathbf{x}}{\partial t} = 0$ and H is autonomous

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}) \Rightarrow \frac{\mathrm{d}x_i}{\mathrm{d}t} = \{x_i, H\} = \{\cdot, H\}x_i \Leftrightarrow$$
$$\Leftrightarrow x_i(t) = e^{t\{\cdot, H\}}x_i(0). \tag{13}$$

Propagator

This will propagate our system a time t

$$\mathbf{x}(t) = e^{t\{\cdot,H\}}\mathbf{x}(0) \tag{14}$$

Oh... also Taylor expansion

$$f(t_0 + \Delta t) = e^{\Delta t \frac{\mathrm{d}}{\mathrm{d}t}} f(t)\,|_{t=t_0}, \tag{15}$$

Symplectic integrator

A set of coordinates are canonical if θ is preserved

A transformation between two canonical coordinates that preserves the Hamiltonian form is a canonical transformation (symplectomorphism)

Symplectic integrators preserves the symplectic form, i.e. $\omega(H^h q, H^h p) = \omega(q, p)$

Symplectic integrator

Numerical flow is called symplectic if

$$(\mathcal{J}\Phi_h)^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{J}\Phi_h = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{16}$$

where the Jacobian matrix is

$$\mathcal{J}\mathbf{F}(\mathbf{x}) = \frac{\partial F_i}{\partial x_j} \qquad \forall\ i,j \in \mathbb{N} : 1 \leq i \leq n, 1 \leq j \leq m. \tag{17}$$

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The actual split

Kepler and perturbation

$$H = H_K + H_I \tag{18}$$

but

$$e^{h\{\cdot, H_K + H_I\}} = e^{h(\{\cdot, H_K\} + \{\cdot, H_I\})} \neq e^{h\{\cdot, H_K\}} e^{h\{\cdot, H_I\}} \tag{19}$$

The actual split

However

$$e^{h(\{\cdot,H_K\}+\{\cdot,H_I\})} \approx e^{h\{\cdot,H_K\}}e^{h\{\cdot,H_I\}} \tag{20}$$

So how does this help?

The actual split

only $e^{h\{\cdot,H_K\}}$: system integrable
only $e^{h\{\cdot,H_I\}}$: system integrable
so

$$\mathbf{x}_K(h) = e^{h\{\cdot,H_K\}}\mathbf{x}(0) \tag{21}$$

$$\mathbf{x}(h) = e^{h\{\cdot,H_I\}}\mathbf{x}_K(h) \tag{22}$$

So we end up with h along one solution and h along the other...

The actual split

So order must matter?

How do you show accuracy?

How do you ensure symplectic structure?

Symmetric $\Phi_h = \Phi_{-h}^{-1}$?

Order?

The numerical flow Φ is of order p if the Taylor expansion match real flow Ψ to term number p.

Consider $H = H_A + H_B$

Order?

$$e^{h(\{\cdot,H_A\}+\{\cdot,H_B\})} = \sum_{i=0}^{\infty} \frac{h^i(\{\cdot, H_A\} + \{\cdot, H_B\})^i}{i!} \tag{23}$$

$$e^{h\{\cdot,H_A\}} e^{h\{\cdot,H_B\}} = \left(\sum_{i=0}^{\infty} \frac{h^i\{\cdot, H_A\}^i}{i!} \right) \left(\sum_{i=0}^{\infty} \frac{h^i\{\cdot, H_B\}^i}{i!} \right) \tag{24}$$

Second order?

By subtracting the two expressions we can easily see that all first order terms vanish

$$
\begin{aligned}
e^{h\{\cdot,H_A\}}e^{h\{\cdot,H_B\}} - e^{h(\{\cdot,H_A\}+\{\cdot,H_B\})} &= \\
= \text{Loads of terms and general dizziness} &= \\
&= \mathcal{O}(h^2) \qquad (25)
\end{aligned}
$$

Second order?

Thus the split

$$e^{h\{\cdot,H_B\}}e^{h\{\cdot,H_A\}} \tag{26}$$

is a first order split (if H_A potential and H_B kinetic it is symplectic Euler integration), however if we instead split

$$e^{\frac{h}{2}\{\cdot,H_{kep}\}}e^{h\{\cdot,H_I\}}e^{\frac{h}{2}\{\cdot,H_{kep}\}} \tag{27}$$

We find the most basic second order split with error $\mathcal{O}(h^3)$

where $e^{\frac{h}{2}\{\cdot,H_{kep}\}}$ drifts along a Kepler orbit
and $e^{h\{\cdot,H_I\}}$ kicks the momentum

Going nuts

Shorter notation: $e^{h\{\cdot,H_A\}} = H_A^h$

$$H_a^{a_1} H_b^{b_1} H_a^{a_2} H_b^{b_2} H_a^{a_3} H_b^{b_3} H_a^{a_4} H_b^{b_4} H_a^{a_5} H_b^{b_4} H_a^{a_4} H_b^{b_3} H_a^{a_3} H_b^{b_2} H_a^{a_2} H_b^{b_1} H_a^{a_1} \tag{28}$$

is a 8 order $H = H_a + H_b$ split

Going nuts

With numerical values of the coefficients as

$$H^h = \left\{ \begin{array}{l} a_1 = 0.03809449742241219545697532230863756534060h \\ a_2 = 0.1452987161169137492940200726606637497442h \\ a_3 = 0.2076276957255412507162056113249882065158h \\ a_4 = 0.4359097036515261592231548624010651844006h \\ a_5 = -0.6538612258327867093807117373907094120024h \\ b_1 = 0.09585888083707521061077150377145884776921h \\ b_2 = 0.2044461531429987806805077839164344779763h \\ b_3 = 0.2170703479789911017143385924306336714532h \\ b_4 = -0.01737538195906509300561788011852699719871h \end{array} \right. \tag{29}$$

Time zone splits

Time to go even more nuts:
Shorter notation
D stands for drift
K for kick and
L for linear drift (barycenter)

Time zone splits

Goal: different timezones, while still preserving symplectic map

One way: find arbitrary sub-splits and construct the integrator

Time zone splits

If we denote K_{ij} as the interaction Hamiltonian between zone i and j we can construct a 3 time zone integrator.

This works due to transition functions to smoothly transfer objects from one time zone to another.

Time zone splits

$$
\begin{aligned}
H^h = & L^{h/8} K_{00}^{h/8} D_0^{h/4} K_{00}^{h/8} L^{h/8} \\
& K_{01}^{h/4} K_{11}^{h/4} D_1^{h/2} K_{11}^{h/4} K_{01}^{h/4} \\
& L^{h/8} K_{00}^{h/8} D_0^{h/4} K_{00}^{h/8} L^{h/8} \\
& K_{02}^{h/2} K_{12}^{h/2} K_{22}^{h/2} D_2^{h} K_{22}^{h/2} K_{12}^{h/2} K_{02}^{h/2} \\
& L^{h/8} K_{00}^{h/8} D_0^{h/4} K_{00}^{h/8} L^{h/8} \\
& K_{01}^{h/4} K_{11}^{h/4} D_1^{h/2} K_{11}^{h/4} K_{01}^{h/4} \\
& L^{h/8} K_{00}^{h/8} D_0^{h/4} K_{00}^{h/8} L^{h/8}.
\end{aligned}
\tag{30}
$$

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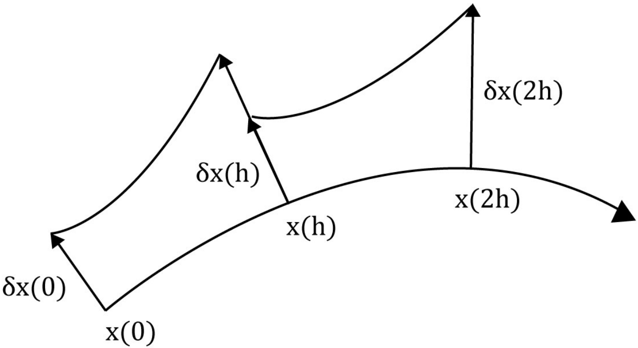
Lyapunov



Lyapunov

I.e. in 1998 sample of Jupiter family comets (JFCs) and near-Earth asteroids (NEAs):
Found to have Lyapunov times between 50 and 150 yr

$$|\delta \mathbf{x}(t)| \approx e^{\lambda t} |\delta \mathbf{x}(0)| \tag{31}$$

where Lyapunov time $T = \lambda^{-1}$ is the time for e^1 divergence to occur.

Lyapunov

If $T = 100$ yr then errors in a 300 yr simulation ≈ 20 times enlarged

Variational flow on the tangent space

The deviation vector $w(t) \in TM$ defined as

$$w(t) = \delta x_i(t) \; \forall \; i \in [1, 2N], \tag{32}$$

variation of differential equation flow $D_x \Phi_t$ with respect to the phase space trajectory

$$D_x \Phi_t \; : \; T_{x(0)} M \mapsto T_{x(t)} M. \tag{33}$$

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The actual research

Mostly clever use of statistics and not that much math (yet)

E.g Output distributions that at infinite sampling are invariants of parts of input distributions

The actual research

Some of the stuff I scribbled yesterday

$$((\chi, \varrho), \tau, \mu) \sim (F, G, T, M), \tag{34}$$

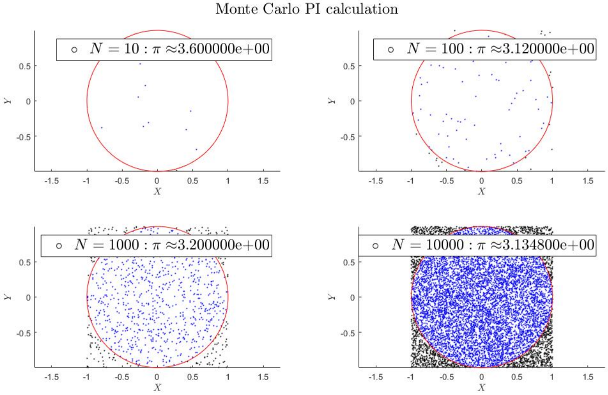
$$(\chi, \varrho) \in M, \tau \in \mathbb{T}, \mu \in \mathbb{M}, \tag{35}$$

$$f \propto \dot{M} \propto r^{-2}. \tag{36}$$

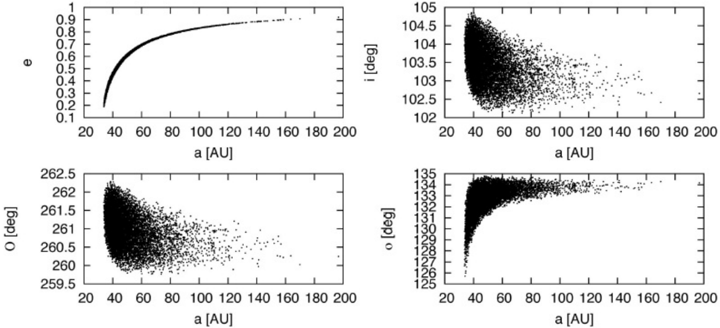
Derived general result that

$$f(\nu) = \frac{(1 + e \cos \nu)^2}{\pi(2 + e^2)}. \tag{37}$$

MC example

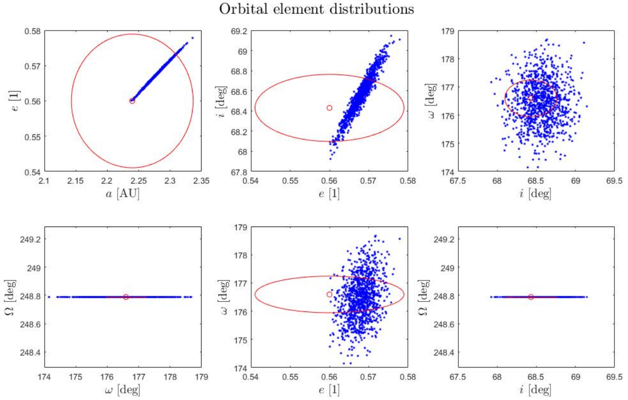

Monte Carlo PI calculation
N = 10 : π ≈3.600000e+00
N = 100 : π ≈3.120000e+00
N = 1000 : π ≈3.200000e+00
N = 10000 : π ≈3.134800e+00
Y
X
0.5
0
-0.5
-1.5
-1
-0.5
0
0.5
1
1.5

Statistical Uncertainty Orbital Clones



Adopted from "OpenOrb: Open-source asteroid orbit computation software including statistical ranging" by GRANVIK et al

Uncertainty example



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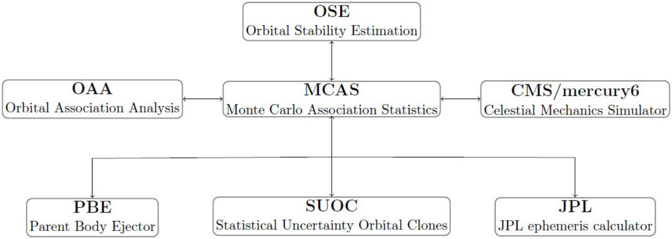
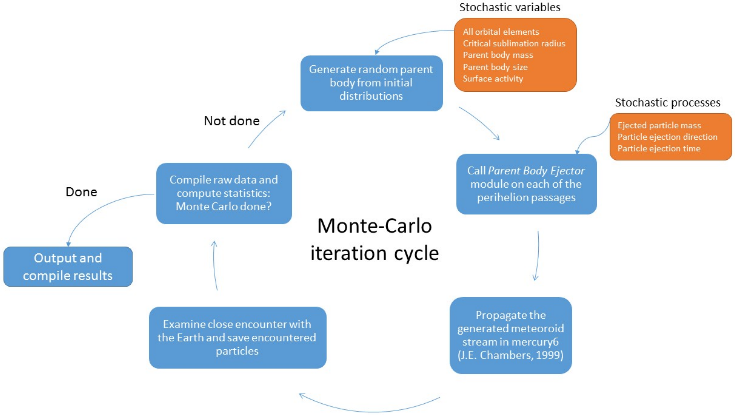
Modular toolbox

MCAS: ~ 14 000 rows
PBE: ~ 7 500 rows
OAA: ~ 4 000 rows

mercury6: ~ 8 000 rows



Program flow



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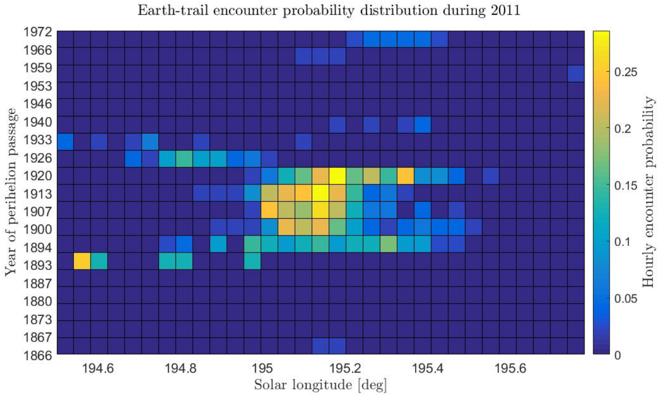
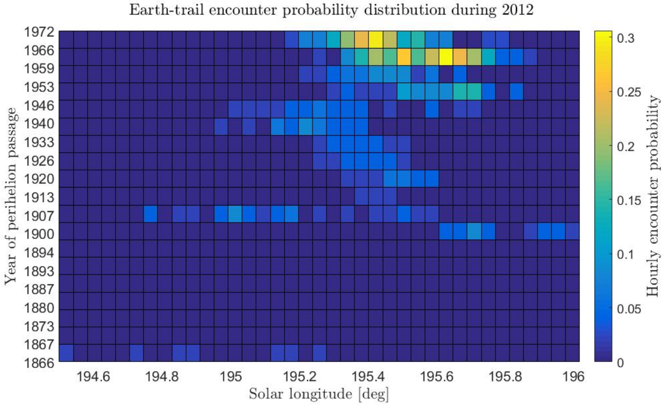
Probability distributions 2011



Probability distributions 2012



Mass difference