Nonlinear Systems
and Their
Remarkable Mathematical Structures
(Vol. 1)

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Contents

Part B: Solution Methods and Solution Structures

Part C: Symmetry Methods for Nonlinear Systems

Part D: Nonlinear Systems in Applications

Preface

The aim of this book is to provide a comprehensive account of the state of the art on the mathematical description of nonlinear systems. The book consists of 20 invited contributions written by leading experts in different aspects of nonlinear systems that include ordinary and partial differential equations, difference equations and q-difference equations, discrete or lattice equations, non-commutative and matrix equations, stochastic equations, as well as supersymmetric equations. The contents is divided into four main parts, namely **Part A**: *Integrable Systems*, **Part B**: *Solution Methods and Solution Structures*, **Part C**: *Symmetry Methods of Nonlinear Systems*, and **Part D**: *Nonlinear Systems in Applications.* Below we give a short description of each contribution.

Part A consists of five contributions, numbered A1 to A5. In this part the authors mainly address the fundamental question of how to detect integrable systems. In **A1** the author *F Calogero* describes how algebraic operations can be applied to study and solve nonlinearly-coupled differential equations. In **A2** the authors *A S Fokas and B Pelloni* give a comprehensive review of the so-called Fokas Method to solve initial-boundary value problems applied to nonlinear partial differential equations on the half-line. In **A3** the authors *B Grammaticos, A Ramani, R Willox and T Mase* describe how to detect discrete integrability by the singularity approach; the singularities of which arise for second order rational mappings. In **A4** the author *J Hietarinta* provides an elementary introduction to discrete or lattice soliton equations, where a multidimensional setting is discussed in some detail. In the last contribution of this section, namely **A5**, the authors *Yu B Suris and M Petrera* discuss new results of the Hahan-Hirota-Kimura discretizations and provide, amongst other things, the integrals of motion.

Part B makes up the largest part of the book and consists of six contributions, numbered B1 to B6. Here the authors describe different methods by which to obtain explicit solutions of nonlinear systems and/or describe the solution structures of the systems. In **B1** the author *O Bihun* discusses a general method to construct isospectral matrices that are defined in terms of the zeros of certain polynomials. It is shown, amongst other things, how this leads to solvable nonlinear first-order ordinary differential systems. In **B2** the authors *R Conte, T W Ng and C Wu* provide a tutorial introduction to methods that were recently developed by the authors in order to find all meromorphic particular solutions of nonintegrable, autonomous, algebraic ordinary differential equations of any order. Some examples in Physics are given. In **B3** the authors *O E Hentosh , Ya A Prykarpatsky, D Blackmore and A Prykarpatski* give a review to the Buhl compatible vector field equation problem by emphasizing its Pfeiffer and Lax–Sato type solutions. They furthermore analyze the related Lie-algebraic structures and integrability of the so-called heavenly equations. An interesting related Lagrange–d'Alembert principle is also discussed in this contribution, as well as other related aspects. In **B4** the author *X B Hu* is concerned with superposition formulae and Bianchi identities that are related to bilinear Bäcklund transformations for nonlinear integrable equations in bilinear form. The author utilizes this to generate soliton solutions, rational solutions as well as some other special solutions to some nonlinear integrable equations. Many examples are provided. In **B5** the author *C Schiebold* constructs $m \times n$-matrix valued solutions for the AKNS system. A complete asymptotic description is given for multiple pole solutions, including wave packets of weakly bound breathers. The collision of vector solitons is also studied. In **B6** the author *C Valls* uses a new method, recently introduced by A Gasull and H Giacomini, to characterize all traveling wave solutions of the Generalized Korteweg-de-Vries-Burgers equation as well as for the Kuramoto-Sivashinsky equation.

In **Part C** the authors concentrate on symmetry methods for nonlinear systems. It consists of five contributions, numbered C1 to C5. In **C1** the authors *M Euler and N Euler* describe the multipotentialisation process by which it is possible to construct nonlocal invariance of certain nonlinear symmetry-integrable evolution equations in $1+1$ dimensions. A complete account of this process for the fifth-order Kupershmidt equation and its hierarchies is given. In **C2** the author *G Gaeta* discusses several aspects of the geometry of vector fields in (Poincaré-Dulac) normal form. This relies substantially on Michel theory. The case, common in Physics, of systems enjoying an *a priori* symmetry is also discussed in some detail. In **C3** the authors *A V Kiselev, A O Krutov and T Wolf* provide an informal discussion of the step-by-step computation of nonlocal recursions for symmetry algebras of nonlinear coupled boson-fermion $N = 1$ supersymmetric systems by using the SsTools environment. In **C4** the author *R Kozlov* discusses Lie point symmetries of Itô stochastic differential equations, which correspond to Lie group transformations of the independent variable (time) and dependent variables that preserve the differential form of the equations and the properties of Brownian motion. In **C5** the authors *M Oberlack, M Wacławczyk and V Grebenev* describe the so-called statistical symmetries of turbulence and discuss their importants in understanding the statistics of turbulence such as intermittency and non-gaussianity.

Part D provides some of the applications of nonlinear systems. This part consists of four contributions, numbered D1 to D4. In **D1** the authors *A Chávez, H Prado and E G Reyes* review their work on integral transforms for ordinary differential equations of infinite order and apply the theory to equations defined with the help of the Riemann zeta function which are of interest in modern theoretical physics. In **D2** the authors *A Constantin and R S Johnson* discuss the rôle of nonlinearity in geostrophic ocean flows on a sphere and point out some of its challenges. In **D3** the authors *A V Osipov and G Söderbacka* give a review of their results on a class of systems of the type n predators - one prey. They also discuss, amongst other things, the extinction and different simple and complicated coexistence of the predators. In **D4** the authors *C Rogers and W K Schief* discuss Ermakov-type systems in nonlinear physics and continuum mechanics. They describe applications in rotating shallow water theory, magnetogasdynamics, multi-layer hydrodynamics and many-body theory. The authors also review their recent work on hybrid Ermakov-Painlevé systems.

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