Some additional notes on ordinary differential equations for M0031M

Norbert Euler

Department of Mathematics, Luleå University of Technology
SE-971 87 Luleå, Sweden

Note: All differential equations referred to below are **ordinary scalar differential equations**, i.e. differential equations in one independent variable, x, and one dependent variable, $y(x)$.

1 The Wronskian: Linear independence of functions and general solutions of linear differential equations

Consider a linear homogeneous ordinary differential equation of order n namely

$$\boxed{p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots p_1(x)y' + p_0(x)y = 0} \tag{1.1}$$

where p_j $(j = 0, 1, 2, \dots, n)$ are real-valued continuous functions given on some domain $\mathcal{D} \subseteq \mathcal{R}$ and $n \geq 1$ with $p_n \neq 0$. Let

$$\{\phi_1(x),\ \phi_2(x), \dots, \phi_s(x)\} \tag{1.2}$$

be a set of solutions of (1.1) in $\mathcal{D}$. That is

$$p_n(x)\phi_j^{(n)} + p_{n-1}(x)\phi_j^{(n-1)} + \cdots p_1(x)\phi_j' + p_0(x)\phi_j = 0, \qquad j = 1, 2, \dots, s. \tag{1.3}$$

Note that every ϕ_j in the set (1.2) are functions which are at least n times continuously differentiable and hence belong to the vector space $C^n(\mathcal{D})$. Then we can state

Theorem 1.1: (Linear Superposition Principle): *Any linear combination of the set of solutions, (1.2), on $\mathcal{D}$, namely*

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_s\phi_s(x), \tag{1.4}$$

is also a solution of (1.1) on $\mathcal{D}$.

Definition 1.1: *The* **general solution** *of an nth-order differential equation on some domain* $\mathcal{D} \subseteq \mathcal{R}$*, is a function,* $\phi(x) \in \mathcal{C}^n(\mathcal{D})$*, which satisfies the differential equation and which contains n arbitrary and independent constants. The domain,* $\mathcal{D}$*, is known as the* **solution domain** *of the differential equation. Solutions which follow from the general solution by choosing values for the arbitrary constant, are known as* **special solutions** *for the differential equation. There may exist solutions outside the domain,* $\mathcal{D}$*, that are not contained in the general solution. Such solutions are known as* **singular solutions** *of the differential equation.*

For the linear differential equation, (1.1), we have

Theorem 1.2: *Let the set of n functions,*

$$\{\phi_1(x),\ \phi_2(x), \dots, \phi_n(x)\}, \tag{1.5}$$

be solutions of (1.1) and be linear independent on $\mathcal{D}$*, then the* **general solution***,* $y(x)$*, of (1.1) on* $\mathcal{D}$ *is given by the linear combination of these n solutions, namely*

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x), \tag{1.6}$$

where $c_1,\ c_2, \dots, c_n$ *are real arbitrary constants.*

To determine whether a set of functions are linearly independent in some domain $\mathcal{D}$ in the vector space $\mathcal{C}^n(\mathcal{D})$, we introduce the so-called Wronskian of the set of functions (1.5).

Definition 1.2: *Let (1.5) be a set of differentiable functions on* $\mathcal{D}$*. The following determinant*

$$W[\phi_1, \phi_2, \dots, \phi_s](x) := \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_s \\ \phi_1' & \phi_2' & \dots & \phi_s' \\ \vdots & \vdots & \dots & \vdots \\ \phi_1^{(s-1)} & \phi_2^{(s-1)} & \dots & \phi_s^{(s-1)} \end{vmatrix} \tag{1.7}$$

is defined as the **Wronskian** *of the set (1.5), where* $W[\phi_1, \phi_2, \dots, \phi_s](x)$ *is also a differentiable function on* $\mathcal{D}$*.*

Example: If $\phi_1(x) = x$ and $\phi_2(x) = \cos x$ in $\mathcal{D} := \mathcal{R}$, then

$$W[\phi_1, \phi_2](x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x. \tag{1.8}$$

Theorem 1.3:

a) Let $S = \{\phi_1(x),\ \phi_2(x), \dots, \phi_n(x)\}$ *be a set of n functions which are all at least n times differentiable on an interval* $\mathcal{D}$*. If the set S is linearly dependent on* $\mathcal{D}$*, then the Wronskian* $W[\phi_1, \dots, \phi_n](x) = 0$ *for all* $x \in \mathcal{D}$*. Moreover, if* $W[\phi_1, \dots, \phi_n](x_0) \neq 0$ *for any choosen point* $x_0 \in \mathcal{D}$*, then the set S is linearly independent on* $\mathcal{D}$*.*

b) Let $S = \{\phi_1(x),\ \phi_2(x), \ldots, \phi_n\}$*, be solutions of (1.1) and let* $W[\phi_1, \ldots, \phi_n](x_0) \neq 0$*, where* x_0 *is any point in* $\mathcal{D}$*. Then the linear combination of*

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x), \tag{1.9}$$

is the general solution of (1.1) on $\mathcal{D}$ *where* $c_1, \ldots, c_n$ *are arbitrary real constants.*

Example: Consider the two exponential functions, namely

$$\phi_1(x) = e^{\alpha_1 x}, \qquad \phi_2(x) = e^{\alpha_2 x}, \tag{1.10}$$

where α_1 and α_2 are any real numbers. We show that ϕ_1 and ϕ_2 are linearly independent on $\mathcal{R}$ if and only if $\alpha_1 \neq \alpha_2$. We evaluate the Wronskian of ϕ_1 and ϕ_2 in the point $x = 0$:

$$W[\phi_1, \phi_2](0) = \left| \begin{matrix} e^{\alpha_1 x} & e^{\alpha_2 x} \\ \alpha_1 e^{\alpha_1 x} & \alpha_2 e^{\alpha_2 x} \end{matrix} \right|_{x=0} = \alpha_2 - \alpha_1.$$

Since $W[\phi_1, \phi_2](x) \neq 0$ in the point $x = 0$ when $\alpha_1 \neq \alpha_2$ it follows by Theorem 1.3(a) that ϕ_1 and ϕ_2 are linearly independent on $\mathcal{R}$ for $\alpha_1 \neq \alpha_2$.

Example: Consider the complex function

$$f(x) = e^{(\alpha + i\beta)x}, \qquad \alpha \in \mathcal{R},\ \beta \in \mathcal{R},\ \ i^2 := -1. \tag{1.11}$$

Since

$$e^{(\alpha + i\beta)x} = e^{\alpha x}\left(\cos\beta x + i \sin\beta x\right) \tag{1.12}$$

we have

$$\phi_1(x) := \mathrm{Re}\,[f(x)] = e^{\alpha x}\cos\beta x, \qquad \phi_2(x) := \mathrm{Im}\,[f(x)] = e^{\alpha x}\sin\beta x. \tag{1.13}$$

It is now easy to show that

$$W[\phi_1,\ \phi_2](0) = \beta. \tag{1.14}$$

Hence, it follows by Theorem 1.3(a) that ϕ_1 and ϕ_2 are linearly independent on $\mathcal{R}$ for $\beta \neq 0$.

Example: One may verify that

$$\phi_1(x) = x^{-2}\cos(3\ln x) \qquad \text{and} \qquad \phi_2(x) = x^{-2}\sin(3\ln x)$$

are solutions of

$$x^2 y'' + 5xy' + 13y = 0 \tag{1.15}$$

on the interval $\mathcal{D} = \{x \in \mathcal{R} :\ x > 0\}$. Now

$$W[\phi_1, \phi_2](x)$$

$$= \left| \begin{matrix} x^{-2}\cos(3\ln x) & x^{-2}\sin(3\ln x) \\ -x^{-4}\left[3x\sin(3\ln x) + 2x\cos(3\ln x)\right] & x^{-4}\left[3x\cos(3\ln x) - 2x\sin(3\ln x)\right] \end{matrix} \right|$$

A convenient point in $\mathcal{D}$ to evaluate the above Wronskian is at $x = 1$. Thus

$$W[\phi_1, \phi_2](1) = \begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} = 3.$$

Since $W[\phi_1, \phi_2](1) \neq 0$ it follows by Theorem 1.3 that ϕ_1 and ϕ_2 are linear independent on $\mathcal{D}$ and that the general solution of (1.15) is given by

$$y(x) = c_1 x^{-2} \cos(3 \ln x) + c_2 x^{-2} \sin(3 \ln x),$$

where c_1 and c_2 are arbitrary constants.

Exercises:

1. Determine whether the following set of functions, $\{f_1,\ f_2,\ f_3\}$, is linearly independent on $\mathcal{D}$, as given below:
   a) $f_1(x) = e^x,\ f_2(x) = e^{2x},\ f_3(x) = e^{3x}, \quad \mathcal{D} := \mathcal{R}$
   b) $f_1(x) = \ln(x),\ f_2(x) = \ln(x^2),\ f_3(x) = e^{3x}, \quad \mathcal{D} := (0, \infty)$
   c) $f_1(x) = e^{\sin x}\ f_2(x) = e^x,\ f_3(x) = \sin x, \quad \mathcal{D} := \mathcal{R}$

2. Use the following set of functions,

   $$f_1(x) = e^{-x}, \quad f_2(x) = e^{3x}, \quad f_3(x) = e^{4x},$$

   to construct the general solution for the equation

   $$y''' - 6y'' + 5y' + 12y = 0.$$

3. Let $\phi_1(x)$ and $\phi_2(x)$ be two solutions on an interval $\mathcal{D}$ of the differential equation

   $$y'' + p(x)y' + q(x)y = 0, \tag{1.16}$$

   where p and q are any given continuous functions on $\mathcal{D}$.

   a) Show that the Wronskian, $W[\phi_1,\ \phi_2](x)$, satisfies the first-order differential eqation

   $$\frac{dW}{dx} + p(x)W = 0. \tag{1.17}$$

   b) Assume that one solution, $\phi_1(x)$, of (1.16) is known. Show now that the second solution, ϕ_2, satisfies the linear first-order differential equation

   $$\phi_2' - \left(\frac{\phi_1'}{\phi_1}\right)\phi_2 = \frac{c_1}{\phi_1} e^{-\int p(x)dx}, \tag{1.18}$$

   where c_1 is an arbitrary constant.
   Hint: Integrate (1.17).

c) Consider the differential equation

$$xy'' - (2x+1)y' + (x+1)y = 0$$

which admits the solution

$$\phi_1(x) = e^x.$$

Find a second solution, ϕ_2, such that ϕ_1 and ϕ_2 are linear independent and give the general solution of this differential equation.
Hint: Make use of the formula which you derived in b).

4. Prove Theorems 1.1, 1.2 and 1.3.

2 Some linearizable first-order differential equations

2.1 The linear first-order equation

The linear first-order differential equation is of the form

$$\boxed{y' + g(x)y = h(x)} \tag{2.1}$$

where g and h are continuous functions on an interval $\mathcal{D} \subseteq \mathcal{R}$. The general solution can then be given in terms of a formula:

Theorem 2.1: *The general solution of (2.1) is*

$$y(x) = e^{-G(x)} \left[\int h(x) e^{G(x)} dx + c \right], \tag{2.2}$$

where c is an arbitrary constant and $G(x)$ is an anti-derivative of $g(x)$, i.e.

$$G(x) = \int g(x) dx. \tag{2.3}$$

Proof: Multiply (2.1) by the integrating factor $e^{G(x)}$: Thus

$$y' e^{G(x)} + g(x) y e^{G(x)} = h(x) e^{G(x)} \qquad \text{or equivalently} \qquad \frac{d}{dx}\left(y e^{G(x)}\right) = h(x) e^{G(x)}.$$

Integration of the previous relation over x leads to (2.2). □

2.2 A general linearizable case

Some classes of nonlinear first-order differential equations can easily be linearized by a suitable change of variables. In particular, we consider the first-order equation

$$\boxed{\frac{df(y)}{dy}\frac{dy}{dx} + f(y)P(x) = Q(x)} \tag{2.4}$$

where $f(y)$ is any differentiable fuction of y and P and Q are continuous functions of x on some domain $\mathcal{D}$.

We show that (2.4) can be linearised by introducing a new dependent variable, v, by the following substitution:

$$\boxed{v(x) = f(y(x))} \tag{2.5}$$

The first derivative of v is

$$\frac{dv}{dx} = \frac{df}{dy}\frac{dy}{dx}, \tag{2.6}$$

so that (2.4) becomes a linear first-order differential equation in v:

$$\frac{dv}{dx} + P(x)v = Q(x). \tag{2.7}$$

With the general solution of (2.7) we obtain the general solution of (2.4) by the relation (2.5).

Example: We linearize the equation

$$\frac{dy}{dx} + 1 = 4e^{-y}\sin x. \tag{2.8}$$

Note that an equivalent form of (2.8) is

$$e^y\frac{dy}{dx} + e^y = 4\sin x$$

so that a suitable new depedent variable is

$$v(x) = e^y, \qquad \text{with} \qquad \frac{dv}{dx} = e^y\frac{dy}{dx}.$$

Equation (2.8) then takes the linear form

$$\frac{dv}{dx} + v = 4\sin x.$$

Exercise:

Find the general solution of the following nonlinear differential equation by a suitable linearization:

$$\sin y\frac{dy}{dx} = \cos x\left(2\cos y - \sin^2 x\right). \tag{2.9}$$

2.3 The Bernoulli equation

The **Bernoulli equation**, named after Jacob Bernoulli (1654 – 1705) who was one of the prominent mathematicians in the Bernoulli family, is a well-known and important special case of (2.4), namely

$$\boxed{\frac{dy}{dx} = f(x)y + g(x)y^n, \qquad n \in \mathcal{R}\backslash\{0,1\}} \tag{2.10}$$

where f and g are given continuous functions on some domain $\mathcal{D}$. Equation (2.10) can be written in the form

$$y^{-n}\frac{dy}{dx} - f(x)y^{1-n} = g(x). \tag{2.11}$$

By comaparing (2.11) with (2.4) we note that (2.10) must be linearizable by the following change of the dependent variable:

$$v(x) = y^{1-n}(x) \qquad \text{so that} \qquad \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}. \tag{2.12}$$

In terms of the dependent variable v, (2.10) then takes the linear form

$$\frac{dv}{dx} - (1-n)f(x)v = (1-n)g(x) \tag{2.13}$$

Exercise:

Find the general solution of the following Bernoulli equations and give the domain for which the solution is valid:

a) $\dfrac{dy}{dx} + y = x\,y^3$

b) $\dfrac{dy}{dx} + \dfrac{2}{x}\,y = e^x\sqrt{y}$

2.4 The Riccati equation

The Riccati equation, named after the Italien mathematician Jacopo Francesco Riccati (1676–1754), is of the form

$$\boxed{\frac{dy}{dx} = f(x)y^2 + g(x)y + h(x)} \tag{2.14}$$

where f, g and h are any give continuous functions on some domain $\mathcal{D}$.

One of the remarkable properties of the Riccati equation, (2.14), is that it can be linearized in a first-order differential equation if any solution of (2.14) is known. In particular, let $\phi(x)$ be a solution on $\mathcal{D}$ and introduce the following change of dependent variable:

$$y(x) = \phi(x) + z(x) \qquad \text{with} \qquad \frac{dy}{dx} = \frac{d\phi}{dx} + \frac{dz}{dx} \tag{2.15}$$

Since $\phi(x)$ is assumed to be a solution of (2.14), i.e.

$$\frac{d\phi}{dx} = f(x)\phi^2 + g(x)\phi + h(x), \tag{2.16}$$

we obain, with (2.15), the following equation in the dependent variable z:

$$\frac{dz}{dx} = [2\phi f(x) + g(x)]\, z + f(x) z^2. \tag{2.17}$$

We recognize (2.17) as a special Bernoulli equation, (2.10), which linearizes by introducing a new dependent variable, $v(x)$, as follows:

$$z(x) = \frac{1}{v(x)} \qquad \text{with} \quad \frac{dz}{dx} = -\frac{1}{v(x)^2}\frac{dv}{dx}. \tag{2.18}$$

Inserting (2.18) in (2.17) we obtain the linear equation

$$\frac{dv}{dx} + [2\phi(x) f(x) + g(x)]\, v = -f(x). \tag{2.19}$$

To sum up: The Riccati equation, (2.14), can be linearized in a first-order differential equation by the following change of dependent variable:

$$y(x) = \phi(x) + \frac{1}{v(x)}, \tag{2.20}$$

where ϕ is any solution of the Riccati equation, (2.14), and v satisfies the linear equation, (2.19). Thus the general solution of (2.14) can be obtained by the general solution, v, of (2.19) and any solution, ϕ, of (2.14) by the relation (2.20).

Exercises:

1. Find the general solution of

   $$y' = -y^2 + \left(2x + \frac{1}{x}\right) y - x^2, \tag{2.21}$$

   where $\phi(x) = x$ is a solution.

2. Show that the Riccati equation, (2.14), linearizes to a second-order linear equation by the following change of depedent variable:

   $$y(x) = -\frac{w'(x)}{w(x)}\frac{1}{f(x)}. \tag{2.22}$$

3. The Riccati equation, (2.14), admits the following nonlinear superposition formula:

   $$y(x) = \frac{c(y_1 - y_2)y_3 - (y_1 - y_3)y_2}{c(y_1 - y_2) - (y_1 - y_3)}, \tag{2.23}$$

where $y_1,\ y_2,\ y_3$ and y_4 are any distinct solutions of the Riccati equation and c is an arbitrary constant. Clearly this superposition formula provides the general solution to the Riccati equation if three distinct solutions are known.

Find now the general solution of equation (2.21) (see Problem 1 above), where (2.21) admits the following three solutions:

$$\begin{aligned} y_1(x) &= x \\ y_2(x) &= \frac{x^3}{x^2-2} \\ y_3(x) &= \frac{x^3+x}{x^2-1}. \end{aligned}$$

3 On particular solutions of nonhomogeneous linear second-order differential equations with constant coefficients

We consider the linear second-order equation

$$\boxed{y'' + py' + qy = f(x)} \tag{3.1}$$

where p and q are given real numbers and f is a given continuous function on some domain $\mathcal{D} \subseteq \mathcal{R}$. With $f \neq 0$, eq. (3.1) is known as a **nonhomogeneous** equation and **homogeneous** otherwise.

Definition 3.1: *Any function, ϕ_p, which satisfies the nonhomogeneous equation (3.1) on an interval $\mathcal{D}$, is known as a particular solution of (3.1) on $\mathcal{D}$.*

Theorem 3.1: *The general solution of (3.1) is of the form*

$$y(x) = \phi_H(x; c_1, c_2) + y_p(x), \tag{3.2}$$

where ϕ_H is the general solution of the associated homogeneous equation (containing two arbitrary constants, c_1 and c_2), i.e. (3.1) with $f(x)$ considered as zero, and y_p is a particular solution of the nonhomogeneous equation (3.1).

The are different methods to find particular solutions for (3.1) and we study here one method, namely the method of undetermined coefficients, which is useful for special functions, $f(x)$, of (3.1).

3.1 The Method of undetermined coefficients to find particular solutions

We consider a method to construct particular solutions for (3.1) for three special cases of the functions $f(x)$ in equation (3.1), namely

Case I: $f(x) = P_n(x)$, where P_n is an nth degree polynomial;

Case II: $f(x) = e^{\alpha_1 x} \cos(\alpha_2 x)\, P_n(x)$ and $f(x) = e^{\alpha_1 x} \sin(\alpha_2 x)\, P_n(x)$, where $\alpha_1 \in \mathcal{R}$, $\alpha_2 \in \mathcal{R}$ and P_n is an nth degree polynomial;

Case III: $f(x) = e^{\alpha x} P_n(x), \qquad \alpha \in \mathcal{R}$. where P_n is also an nth degree polynomial.

We now study the above three cases in detail.

3.1.1 The case $f(x) = P_n(x)$:

Case I: Consider

$$\boxed{y'' + py' + qy = P_n(x)} \tag{3.3}$$

where P_n is an nth-degree polynomial function, i.e.,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0. \tag{3.4}$$

Here $a_j,\ j = 0, 1, \ldots, n$ are given real coefficients and n is a given natural number. We can now make Ansätze to find particular solutions of (3.3). We need to destinguish between three different Ansätze:

Case Ia: Let $\boxed{q \neq 0}$. The Ansatz for a particular solution of (3.3) is then

$$y_p(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0 := Q_n(x), \tag{3.5}$$

where the real constants, $A_j,\ j = 0, 1, \ldots, n$, are to be determined when (3.5) is inserted in (3.3).

Case Ib: Let $\boxed{q = 0 \text{ and } p \neq 0}$. The Ansatz for a particular solution of (3.3) is then

$$y_p(x) = x Q_n(x), \tag{3.6}$$

where $Q(x)$ is given by (3.6).

Case Ic: Let $\boxed{q = 0 \text{ and } p = 0}$. The Ansatz for a particular solution of (3.3) is then

$$y_p(x) = x^2 Q_n(x), \tag{3.7}$$

where $Q(x)$ is given by (3.6).

Example: We find the general solution of

$$y'' + 4y = 8x^2. \tag{3.8}$$

The general solution, ϕ_H, of the associated homogeneous equation

$$y'' + 4y = 0 \tag{3.9}$$

is

$$\phi_H(x) = c_1 \cos(2x) + c_2 \sin(2x). \tag{3.10}$$

For the particular solution we use the Ansatz proposed in Case I a:

$$y_p(x) = A_2 x^2 + A_1 x + A_0 \tag{3.11}$$

so that

$$\begin{aligned} y_p'(x) &= 2A_2 x + A_1 \\ p_p''(x) &= 2A_2. \end{aligned}$$

Inserting the above into (3.8) we obtain

$$2A_2 + 4\left(A_2 x^2 + A_1 x + A_0\right) = 8x^2. \tag{3.12}$$

Equating coefficients of $x^2,\ x$ and 1 leads to a set of linear algebraic equations in $A_2,\ A_1$ and A_0:

$$\begin{aligned} x^2 : &\qquad 4A_2 = 8 \\ x^1 : &\qquad 4A_1 = 0 \\ 1\ \ : &\qquad 2A_2 + 4A_0 = 0, \end{aligned}$$

so that $A_0 = -1,\ A_1 = 0,\ A_2 = 2$. Hence a particular solution for (3.8) is

$$y_p(x) = 2x^2 - 1. \tag{3.13}$$

The general solution of (3.8) is therefore

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + 2x^2 - 1. \tag{3.14}$$

3.1.2 The case $f(x) = e^{\alpha_1 x}\cos(\alpha_2 x)\,P_n(x)$ and $f(x) = e^{\alpha_1 x}\sin(\alpha_2 x)\,P_n(x)$, $\alpha_1, \alpha_2 \in \mathcal{R}$

Let $y_c(x)$ denote a complex function, such that

$$y_c(x) = u(x) + iv(x), \tag{3.15}$$

where u and v are real differentiable functions on some domain $\mathcal{D} \subseteq \mathcal{R}$. Then

$$\begin{aligned} y_c'(x) &= u'(x) + iv'(x), \\ y_c''(x) &= u''(x) + iv''(x). \end{aligned}$$

Furthermore we recall

$$e^{(\alpha_1 + i\alpha_2)x} = e^{\alpha_1 x}\left(\cos(\alpha_2 x) + i\sin(\alpha_2 x)\right), \tag{3.16}$$

where α_1 and α_2 are any real numbers.

Case II: We now consider the following linear **complex differential equation** with depedent variable $y_c(x)$:

$$\boxed{y_c'' + py_c' + qy_c = e^{\alpha x} P_n(x), \qquad \alpha := \alpha_1 + i\alpha_2, \quad \alpha_1 \in \mathcal{R},\ \alpha_2 \in \mathcal{R}} \tag{3.17}$$

where P_n is an nth-degree polynomial function, i.e.,

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{3.18}$$

with $a_j,\ j = 0, 1, \ldots, n$ real coefficients and n a natural number. Here α is a complex number, with $\alpha = \alpha_1 + i\alpha_2$. By (3.15) and (3.16), equation (3.17) takes the form

$$u'' + iv'' + p(u' + iv') + q(u + iv) = e^{\alpha_1 x}\cos(\alpha_2 x)\, P_n(x) + ie^{\alpha_1 x}\sin(\alpha_2 x)\, P_n(x) \tag{3.19}$$

Comparing the real and imaginary parts of (3.19) we obtain the following two real non-homogeneous differential equations in u and v, respectively:

$$\boxed{u'' + pu' + qu = e^{\alpha_1 x}\cos(\alpha_2 x)\, P_n(x)} \tag{3.20}$$

and

$$\boxed{v'' + pv' + qv = e^{\alpha_1 x}\sin(\alpha_2 x)\, P_n(x)} \tag{3.21}$$

We now state the following

Proposition 3.1: *A convenient Ansatz for a complex particular solution, y_{cp}, of (3.17) is*

$$\boxed{y_{cp}(x) = e^{\alpha x} w_c(x)} \tag{3.22}$$

where w_c is any complex solution of the equation

$$\boxed{w_c'' + (2\alpha + p)w_c' + (\alpha^2 + \alpha p + q)w_c = P_n(x)} \tag{3.23}$$

A particular real solution, u_p, for (3.20) is then given by the real part of y_{cp}, i.e.

$$u_p(x) = Re\,[y_{cp}], \tag{3.24}$$

whereas a particular real solution, v_p, of (3.21) is give by the imaginary part of y_{cp}, i.e.

$$v_p(x) = Im\,[y_{cp}]. \tag{3.25}$$

Proof: Differentiating the Ansatz (3.22) we obtain

$$y_{cp}'(x) = \alpha e^{\alpha x} w_c(x) + e^{\alpha x} w_c'(x)$$

$$y_{cp}''(x) = \alpha^2 e^{\alpha x} w_c(x) + 2\alpha e^{\alpha x} w_c'(x) + e^{\alpha x} w_c''(x).$$

Inserting (3.22) and the above derivatives, y'_{cp} and y''_{cp}, in (3.17) leads to condition (3.23).

$\square$

To find a solution, w_c, of (3.23), i.e.

$$w''_c + (2\alpha + p)w'_c + (\alpha^2 + \alpha p + q)w_c = P_n(x),$$

we make use the same type of Ansätze as listed in Case I, since the nonhomogeneous part of (3.23) is a polynomial function, albeit we now need to evaluate complex coefficients in the polynomial Ansatz $Q(x)$. The folowing three cases follow:

Case IIa: Let $\boxed{\alpha^2 + \alpha p + q \neq 0}$. An Ansatz for a solution, w_c, of (3.23) is then

$$w_c(x) = B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0 := S_n(x), \tag{3.26}$$

where B_j $(j = 0, 1, \ldots, n)$ are **complex constants** which need to be determined for this Ansatz.

Case IIb: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p \neq 0}$. An Ansatz for a solution, w_c, of (3.23) is then

$$w_c(x) = xS_n(x), \tag{3.27}$$

where $S_n(x)$ is given by (3.26).

Case IIc: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p = 0}$. An Ansatz for a solution, w_c, of (3.23) is then

$$w_c(x) = x^2 S_n(x), \tag{3.28}$$

where $S_n(x)$ is given by (3.26).

Example: We find a particular solution for the differential equation

$$y'' + 4y = (10x - 1)e^x \cos x. \tag{3.29}$$

Since

$$\text{Re}\left[(10x - 1)e^{(1+i)x}\right] = (10x - 1)e^x \cos x \tag{3.30}$$

we need to consider the complex equation

$$y''_c + 4y_c = (10x - 1)e^{(1+i)x} \tag{3.31}$$

and construct a complex particular solution, y_{cp}, by the Anstaz

$$y_{cp} = e^{(1+i)x}w_c(x). \tag{3.32}$$

A real particular solution, $y_p(x)$, of (3.29) then follows by

$$y_p(x) = \text{Re}\,[y_{cp}]. \tag{3.33}$$

Differentiating the Anstaz (3.33) we obtain

$$\begin{aligned} y'_{cp} &= (1+i)e^{(1+i)x}w_c + e^{(1+i)x}w'_c \\ y''_{cp} &= (1+i)^2e^{(1+i)x}w_c + 2(1+i)e^{(1+i)x}w'_c + e^{(1+i)x}w''_c \end{aligned}$$

and the condition on w_c becomes

$$w''_c + 2(1+i)w'_c + (4+2i)w_c = 10x - 1. \tag{3.34}$$

An appropriate Ansatz for (3.34) is provided by Case IIa, i.e.

$$w_c(x) = B_1x + B_0, \qquad B_0 \in \mathcal{C},\ B_1 \in \mathcal{C}. \tag{3.35}$$

Equation (3.34) then takes the form

$$2(1+i)B_1 + (4+2i)(B_1x + B_0) = 10x - 1 \tag{3.36}$$

and equating coefficients of x and 1 leads to

$$(4+2i)B_1 = 10, \qquad 2(1+i)B_1 + (4+2i)B_0 = -1.$$

We find

$$B_0 = -\frac{8}{5} + \frac{3}{10}i, \qquad B_1 = 2 - i.$$

Thus a complex solution for (3.34) is

$$w_c(x) = (2-i)x - \frac{8}{5} + \frac{3}{10}i \tag{3.37}$$

and a complex particular solution of (3.31) is

$$\begin{aligned} y_{cp}(x) &= e^{(1+i)x}\left[(2-i)x - \frac{8}{5} + \frac{3}{10}i\right] \\ &= e^x\left[\left(2x - \frac{8}{5}\right)\cos x + \left(x - \frac{3}{10}\right)\sin x\right] \\ &\quad + ie^x\left[\left(-x + \frac{3}{10}\right)\cos x + \left(2x - \frac{8}{5}\right)\sin x\right] \end{aligned} \tag{3.38}$$

A real particular solution, $y_p(x)$, of (3.29) is then

$$y_p(x) = \mathrm{Re}\left[y_{cp}\right] = \left(2x - \frac{8}{5}\right)e^x\cos x + \left(x - \frac{3}{10}\right)e^x\sin x. \tag{3.39}$$

3.1.3 The case $f(x) = e^{\alpha x} P_n(x), \quad \alpha \in \mathcal{R}$

An obvious special case of (3.17) is when $\alpha \in \mathcal{R}$, which brings us to

Case III: Consider the equation

$$\boxed{y'' + py' + qy = e^{\alpha x} P_n(x), \qquad \alpha \in \mathcal{R}} \tag{3.40}$$

where P_n is an nth-degree polynomial function with real coefficients. Now $y(x)$ is a real function and the same Ansatz, (3.22), for a particular solution, $y_p(x)$, for (3.40) is valid, namely

$$y_p(x) = e^{\alpha x}\, w(x) \tag{3.41}$$

with the same condition on w as given by (3.23), namely

$$\boxed{w'' + (2\alpha + p)w' + (\alpha^2 + \alpha p + q)w = P_n(x)} \tag{3.42}$$

To find a solution, $w(x)$, for (3.42) we distinguish between three cases:

Case IIIa: Let $\boxed{\alpha^2 + \alpha p + q \neq 0}$. An Ansatz for a solution, w, of (3.42) is then

$$w(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0 := Q_n(x), \tag{3.43}$$

where A_j $(j = 0, 1, \ldots, n)$ are **real constants** which need to be determined for the Ansatz.

Case IIIb: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p \neq 0}$. An Ansatz for a solution, w, of (3.42) is then

$$w(x) = x Q_n(x), \tag{3.44}$$

where $Q_n(x)$ is given by (3.43).

Case IIIc: Let $\boxed{\alpha^2 + \alpha p + q = 0 \text{ and } 2\alpha + p = 0}$. An Ansatz for a solution, w, of (3.42) is then

$$w(x) = x^2 Q_n(x), \tag{3.45}$$

where $Q_n(x)$ is given by (3.43).

Example: We find a particular solution for

$$y'' - 9y = 8x^3 e^x.$$

With the Anstaz (3.41), $\alpha = 3$ and using Case IIIa, we obtain the particular solution

$$y_p(x) = (A_0 + A_1 x + A_2 x^2 + A_3 x^3) e^x,$$

where

$$\begin{aligned} &A_0 = -\frac{15}{32}, \quad A_1 = -\frac{9}{8}, \\ &A_2 = -\frac{3}{4}, \quad A_3 = -1. \end{aligned}$$

Exercises:

Find particular solutions of the following differential equations:

a) $y'' + 2y' + y = xe^x \cos x$.

b) $y'' + 4y = \sin 2x$.

c) $y'' - 6y' + 9y = \left(3x^7 - 5x^4\right) e^{3x}$.