SINGULAR DYNAMICS FOR THE HAMILTON–JACOBI EQUATION

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1. Introduction and background

This talk is devoted to the spreading of singularities for

$$\mathcal{H}(x, S(x), \nabla S(x)) = 0 \qquad \text{in } \Omega \subseteq \mathbb{R}^N \tag{1}$$

where the Hamiltonian $\mathcal{H}(x, r, p)$ is a convex function of $p \in \mathbb{R}^N$.

- (1) arises in optimal control;
- Formation of singularities or shocks;
- Typically, viscosity solutions u of (1) are semiconcave functions, i.e., satisfy condition (A):

(A) For any compact convex subset $\mathcal{K}$ of Ω there exists a constant $c = c_{\mathcal{K}} \geq 0$ such that

$$S(x) - c|x|^2/2$$

is concave in $\mathcal{K}$.

Definition 1. The *singular set* of S consists of all points where S fails to be differentiable. It is denoted by Σ.

1.1. **An elementary example.** Consider the inviscid Burgers (Poisson, Airy, Challis, Stokes) equation

$$u_t + uu_x = 0 \quad \text{or} \quad u_t + (u^2/2)_x = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R},$$

supplemented with the initial condition

$$u(0,x) = u_0(x).$$

With $S(t,x) = \int^x u(t,\xi)\,d\xi$ and $S_0(x) = \int^x u_0(\xi)\,d\xi$ we find that

$$S_t + \frac{1}{2}(S_x)^2 = 0.$$

The solution u is constant along each characteristic

$$\boldsymbol{x}(t,y) = y + tu_0(y),$$

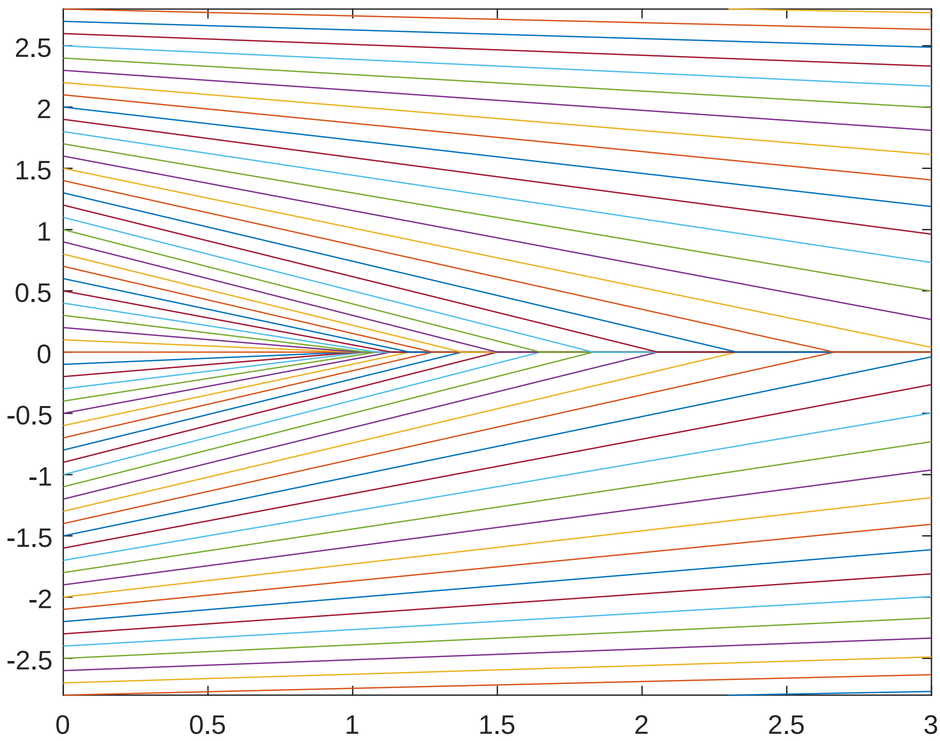
i.e., $u(t,\boldsymbol{x}(t,y)) = u_0(y)$. If u_0 is nondecreasing (i.e., S_0 is convex) then the characteristics do not intersect. Otherwise, characteristics cross and u ceases to exist as a classical solution.

Invertibility of $x = \boldsymbol{x}(t,y)$. If, in $(0,T) \times \mathbb{R}$,

$$\boldsymbol{x}_y(t,y) = 1 + tu_0'(y) = 1 + tS_0''(y) > 0,$$

then the Lagrangian coordinate y is a smooth function of the Eulerian coordinate x (or of (t,x)): $x = \boldsymbol{x}(t,y) \Leftrightarrow y = \boldsymbol{y}(t,x)$ and $u(t,x) = u_0(\boldsymbol{y}(t,x))$ is a classical solution in $(0,T) \times \mathbb{R}$.

Shocks are generated at conjugate points corresponding to $\boldsymbol{x}_y(t,y) = 0$.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Concepts of weak solution: Entropy solution and viscosity solution, respectively. The viscosity solution S is given by

$$S(t,x) = \inf_y \left(S_0(y) + \frac{1}{2t}(x-y)^2 \right)$$

while the entropy solution u derives from S as

$$u(t,x) = S_x(t,x) = \frac{x - y(t,x)}{t} = S_0'(y(t,x)).$$

By expansion,

$$tS(t,x) = \frac{x^2}{2} + \inf_y(-xy + ty^2/2)$$

whence

$$tS(t,x) - \frac{x^2}{2} = \text{a concave function of } (t,x).$$

1.2. **Organization.** I will give an overview focusing on the propagation of singularities along **arcs**, which are likewise **generalized or strong characteristics.**

Outline.

- Generalized characteristics and singular dynamics for
$$S_t + H(x, \nabla S) = 0, \quad t > 0,\ x \in \mathbb{R}^n,$$
$$p_t + H(x, p)_x = 0 \quad t > 0,\ x \in \mathbb{R},$$
$$\mathcal{H}(x, S(x), \nabla S(x)) = 0, \quad x \in \mathbb{R}^N.$$
- Singular dynamics through excess action.

1.3. A classical Cauchy problem. Consider

$$S_t + H(x, \nabla S) = 0 \quad \text{in } Q = (0,\infty) \times \mathbb{R}^n, \tag{2}$$

$$S(0,x) = S_0(x) \quad \text{in } \mathbb{R}^n. \tag{3}$$

(H1) The Lagrangian $L \in C^k(\mathbb{R}^n \times \mathbb{R}^n)$; $\nabla_v^2 L(x,v) > 0$ and $L(x,v) \geq \ell(|v|)$ where $\ell(s)/s \to \infty$ as $s \to \infty$; the Hamiltonian H is the Legendre–Fenchel transform of L, i.e.,

$$H(x,p) = \max_{v \in \mathbb{R}^n} (\langle p, v \rangle - L(x,v)).$$

(H2) The initial function $S_0 \in C^k(\mathbb{R}^n)$ and

$$S_0(x) \geq -C(1 + |x|).$$

One notes that

$$\text{(H1)} \;\Rightarrow\; H \in C^k \text{ and } \nabla_p^2 H > 0.$$

1.3.1. *The value or action function.* We consider the action functional

$$J(\boldsymbol{x}; 0, t) = S_0(\boldsymbol{x}(0)) + \int_0^t L(\boldsymbol{x}(s), \dot{\boldsymbol{x}}(s))\, ds$$

defined on

$$\mathcal{A}(t, x) = \{\boldsymbol{x} \in \mathrm{AC}([0, t]; \mathbb{R}^n) \colon \boldsymbol{x}(t) = x\}.$$

Then, the *action function* is defined by

$$S(t, x) = \inf_{\boldsymbol{x} \in \mathcal{A}(t,x)} J(\boldsymbol{x}; 0, t), \qquad (t, x) \in Q. \tag{4}$$

In general, S fails to be differentiable everywhere yet it is locally semi-concave throughout $\overline{Q}$. It is a viscosity solution of

$$S_t + H(x, \nabla S) = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n,$$
$$S(0, x) = S_0(x) \quad \text{in } \mathbb{R}^n.$$

A point $(t, x) \in Q$ belongs to Σ if and only if there exist two distinct minimizing arcs in (20).

1.3.2. *Conjugate points.* For each $y \in \mathbb{R}^n$, let $(\boldsymbol{x}(t, y), \boldsymbol{p}(t, y))$ denote the solution of Hamilton's system

$$\dot{\boldsymbol{x}}(t) = \nabla_p H(\boldsymbol{x}(t), \boldsymbol{p}(t)), \tag{5}$$

$$\dot{\boldsymbol{p}}(t) = -\nabla_x H(\boldsymbol{x}(t), \boldsymbol{p}(t)), \tag{6}$$

subject to the initial condition

$$(\boldsymbol{x}(0), \boldsymbol{p}(0)) = (y, \nabla S_0(y)).$$

An arc $\boldsymbol{x} \in \mathcal{A}(t, x)$ is an extremal precisely if $\boldsymbol{x} = \boldsymbol{x}(\cdot, y)$ for some initial point $y \in \mathbb{R}^n$.

Definition 2. A point $(t, x) \in Q$ is called *conjugate* if there exists a point $y \in \mathbb{R}^n$ such that $\boldsymbol{x}(t, y) = x$, $\boldsymbol{x}(\cdot, y)$ is a minimizer for (20), and

$$\det \nabla_y \boldsymbol{x}(t, y) = 0.$$

The set of conjugate points is denoted by Γ.

1.3.3. *Necessary conditions.* Every minimizing curve $\boldsymbol{x}(\cdot)$ for (20) is an extremal such that $(s, \boldsymbol{x}(s)) \notin \Sigma \cup \Gamma$ for any $s \in (0, t)$.

1.3.4. *Classical characteristic.* Consider the solution $(\boldsymbol{x}(t,y), \boldsymbol{p}(t,y))$ of Hamilton's system

$$\dot{\boldsymbol{x}}(t) = \nabla_p H(\boldsymbol{x}(t), \boldsymbol{p}(t)), \tag{7}$$

$$\dot{\boldsymbol{p}}(t) = -\nabla_x H(\boldsymbol{x}(t), \boldsymbol{p}(t)), \tag{8}$$

subject to the initial condition

$$(\boldsymbol{x}(0), \boldsymbol{p}(0)) = (y, \nabla S_0(y)).$$

Then, $\boldsymbol{x}(t)$ is a minimizer of Lagrangian action,

$$\boldsymbol{p}(t) = \nabla S(t, \boldsymbol{x}(t)),$$

and

$$\dot{\boldsymbol{x}}(t) = \nabla_p H(\boldsymbol{x}(t), \nabla S(t, \boldsymbol{x}(t)))$$

as long as $(t, \boldsymbol{x}(t)) \notin \Sigma$.

1.3.5. *Generalized characteristic.*

Definition 3. A *generalized characteristic* in an interval $I \subseteq [0, \infty)$ is a locally Lipschitz continuous curve $\boldsymbol{X} \colon I \to \mathbb{R}^n$ such that the differential inclusion

$$\dot{\boldsymbol{X}}(t) \in \operatorname{co} \nabla_p H(t, \boldsymbol{X}(t), \nabla^+ S(t, \boldsymbol{X}(t))) \tag{9}$$

holds for almost all $t \in I$.

1.3.6. *The regularity of the value function.* Some basic properties are collected in

Theorem 1. *Let (H1) and (H2) be fulfilled with $k \geq 2$. Then the following assertions hold.*

(i) $\overline{\Sigma} = \Sigma \cup \Gamma \subset Q$ *and* $S \in C^k(\overline{Q} \setminus \overline{\Sigma})$.

(ii) (Blowup of second derivatives.) *Let* $(\bar{t}, \bar{x}) \in \Gamma$ *and let* $\boldsymbol{x}(\cdot, \bar{y})$ *be a minimizing arc for* $(\bar{t}, \bar{x})$ *such that* $\det \nabla_y \boldsymbol{x}(\bar{t}, \bar{y}) = 0$. *Then*

$$\|\nabla_x^2 S(t, \boldsymbol{x}(t, \bar{y}))\| \to \infty \quad \text{as } t \to \bar{t}^-.$$

1.3.7. *Shock generation.* Points in Γ are shock generation points: given $(t_0, x_0) \in \Gamma$, there exists $M > 0$ such that, for every small $\varepsilon > 0$ there is some x_ε such that $|x_\varepsilon - x_0| < \varepsilon M$ and $(t_0 + \varepsilon, x_\varepsilon) \in \Sigma$.

A Hausdorff estimate:

Theorem 2 (Cannarsa–Mennucci–Sinestrari [26] (1997))**.** *If (H1) and (H2) hold, with $k \geq 2$, then*

$$\mathcal{H}^{n-1+2/(k-1)}(\Gamma \setminus \Sigma) = 0.$$

In particular, if $H \in C^\infty$ and $u_0 \in C^\infty$ then

$$\mathcal{H}\text{-}\dim(\Gamma \setminus \Sigma) \leq n - 1.$$

Remark 1. Article [18] gives an example with C^∞ data such that $\Gamma \setminus \Sigma$ includes **uncountably** many affine subspaces of dimension $n - 1$.

Remark 2. For $n = 1$ and a given $H = H(p)$ from $C^2(\mathbb{R})$ satisfying (A), Li has given an example of an initial function $S_0 \in C^1(\mathbb{R})$ with bounded derivative such that almost every point of the line $t = 0$ is a shock generation point [38].

1.4. **Scalar conservation laws.** Assuming (H1) and (H2) with $n = 1$ (i.e., $x \in \mathbb{R}$) and $k \geq 2$, a function $p(t, x)$ is an admissible or entropy solution [in particular, $p \in L^\infty_{\mathrm{loc}}$ and $p(t, x-) \geq p(t, x+)$ for a.e. t and all x] of

$$p_t + H(x, p)_x = 0 \tag{10}$$

if and only if $p = S_x$ for some viscosity solution S of

$$S_t + H(x, S_x) = 0. \tag{11}$$

1.4.1. *Generalized characteristics.* A locally Lipschitz continuous curve $x = \boldsymbol{X}(t)$ defined for $\tau \leq t < \infty$ is a *generalized characteristic* if

$$\dot{\boldsymbol{X}}(t) \in [H_p(\boldsymbol{X}(t), S_x(t, \boldsymbol{X}(t)+)), \\ H_p(\boldsymbol{X}(t), S_x(t, \boldsymbol{X}(t)-))] \quad \text{a.e. } t \in (\tau, \infty)$$

1.4.2. *Dafermos' description of the shock set.* The singular set Σ is included in the union $\mathcal{S}$ of a family of shock curves

$$\{(t,x)\colon t\in[\tau_j,\infty),\ x=\boldsymbol{X}_j(t)\},\quad \tau_j>0,$$
$$u_x(t,\boldsymbol{X}_j(t)-)>u_x(t,\boldsymbol{X}_j(t)+),\quad t\in(\tau_j,\infty).$$

- $\boldsymbol{X}_j$ is a generalized characteristic in $[\tau_j,\infty)$;
- $(t,\boldsymbol{X}_j(t))\in\Sigma$ when $t\in(\tau_j,\infty)$;
- $(\tau_j,\boldsymbol{X}_j(\tau_j))$ is a *shock generation point*; it belongs to Γ and may belong to Σ.

Conversely, each element of $\Gamma\setminus\Sigma$ is a shock generation point [32, Sect. 5]. A shock generation point may be singular in which case it is the focus of a centered compression wave.

Shock curves merge in general.

2. Sub- and superdifferential

2.1. **Generalized differentiation.** Let

$$f\colon \Omega \to \mathbb{R}.$$

2.1.1. *The superdifferential.* $\partial^+ f\colon \Omega \rightrightarrows \mathbb{R}^N$ is defined by: $p \in \partial^+ f(x)$ if

$$\limsup_{|h|\to 0} \frac{f(x+h) - f(x) - p\cdot h}{|h|} \le 0. \tag{12}$$

Under (A), $\partial^+ u(x)$ is a compact convex nonempty subset of $\mathbb{R}^N$.

2.1.2. *A representation of the superdifferential.* We recall p is a *limiting gradient* of f at x if $\exists x_j \neq x$ such that f is differentiable at x_j and

$$\lim_{j\to\infty} (x_j, \nabla f(x_j)) = (x, p).$$

Let $\overline{\nabla} f(x)$ denote the set of all limiting gradients. If f is semiconcave, then

$$\partial^+ u(x) = \operatorname{co} \overline{\nabla} u(x).$$

2.1.3. *The subdifferential.* $\partial^- f\colon \Omega \rightrightarrows \mathbb{R}^N$ is obtained by replacing, in (12), "lim sup" and "≤" by "lim inf" and "≥," respectively.

3. A general Hamilton–Jacobi equation

We return to (1), i.e., to

$$\mathcal{H}(x, u(x), \nabla u(x)) = 0 \qquad \text{in } \Omega \subseteq \mathbb{R}^N$$

assuming

- **(A)** For any compact convex subset $\mathcal{K}$ of Ω there exists a constant $c = c_{\mathcal{K}} \geq 0$ such that $u(x) - c|x|^2/2$ is concave in $\mathcal{K}$;
- **(B0)** $\mathcal{H} \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^N)$;
- **(B1)** For every $(x, r) \in \Omega \times \mathbb{R}$, $\mathcal{H}(x, r, \cdot)$ is a convex function whose α-level set
$$\{p \in \mathbb{R}^N : \mathcal{H}(x, r, p) = \alpha\}$$
contains a line segment for no $\alpha \in \mathbb{R}$.

3.1. **Viscosity solution.** $f \in C(\Omega)$ is a *viscosity solution* of (1) if, and only if, for every $x \in \Omega$,

$$\mathcal{H}(x, f(x), p) \leq 0 \qquad \text{if } p \in \partial^+ f(x), \text{ and}$$
$$\mathcal{H}(x, f(x), p) \geq 0 \qquad \text{if } p \in \partial^- f(x).$$

Under (A) and (B1), u is a viscosity solution of (1) if and only if

$$\mathcal{H}(x, u(x), \nabla u(x)) = 0 \qquad \text{for a.e. } x \in \Omega.$$

3.2. **Generalized characteristics.**

Definition 4. A Lipschitz continuous arc

$$\boldsymbol{X} \colon [0, T) \to \Omega$$

is called a *generalized characteristic* if it satisfies the differential inclusion

$$\dot{\boldsymbol{X}}(t) \in \operatorname{co} \nabla_p \mathcal{H}\left(\boldsymbol{X}(t), u(\boldsymbol{X}(t)), \partial^+ u(\boldsymbol{X}(t))\right) \quad \text{for a.e. } t \in [0, T). \tag{13}$$

3.2.1. *Remark.* Definition 4 is very weak as it involves two convexification operations enlarging its right-hand side: an outer convexification as well as an inner one appearing implicitly through $\partial^+ u$. As a consequence, there may exist more than one generalized characteristic emanating from a singular point [29, 43].

3.3. **An energy minimal selection of the superdifferential.** A pointwise "energy minimal" selection $\partial^+_{\mathcal{H}} u$ of $\partial^+ u$.

Definition 5. Assuming (A) and (B1), let $\partial^+_{\mathcal{H}} u(x)$ be the unique minimizer of $\mathcal{H}(x, u(x), \cdot)$ over $\partial^+ u(x)$, i.e.,

$$\partial^+_{\mathcal{H}} u(x) \in \partial^+ u(x), \quad \text{and}$$
$$\mathcal{H}\left(x, u(x), \partial^+_{\mathcal{H}} u(x)\right) \leq \mathcal{H}(x, u(x), p)$$
$$\text{for all } p \in \partial^+ u(x).$$

3.3.1. *An observation.* Assuming (A), (B0), (B1), we define

$$M(x) = \mathcal{H}(x, u(x), \partial^+_{\mathcal{H}} u(x)) = \min_{p \in \partial^+ u(x)} \mathcal{H}(x, u(x), p).$$

Then,

$$M(x) \begin{cases} < 0 & \text{if } x \in \Sigma, \\ = 0 & \text{if } x \notin \Sigma, \end{cases}$$

provided that $\mathcal{H}(x, u(x), \nabla u(x)) = 0$ in the viscosity sense.

3.4. **Propagation of singularities along generalized characteristics.**

Theorem 3 (Cannarsa–Yu [29]). *Suppose that u and $\mathcal{H}$ satisfy conditions (A), (B0) and (B1). Let $x_0 \in \Omega$ be a given point. Then there exists a generalized characteristic $\boldsymbol{X}\colon [0,T) \to \Omega$, i.e.,*

$$\dot{\boldsymbol{X}}(t) \in \operatorname{co} \nabla_p \mathcal{H}\left(\boldsymbol{X}(t), u(\boldsymbol{X}(t)), \partial^+ u(\boldsymbol{X}(t))\right) \qquad \text{for a.e. } t \in [0,T),$$

with $\boldsymbol{X}(0) = x_0$ such that

$$\dot{\boldsymbol{X}}_+(0) = \nabla_p \mathcal{H}\left(\boldsymbol{X}(0), u(\boldsymbol{X}(0)), \partial^+_{\mathcal{H}} u(\boldsymbol{X}(0))\right) \tag{14}$$

and

$$\lim_{t \downarrow 0} \operatorname*{ess\,sup}_{s \in (0,t)} |\dot{\boldsymbol{X}}_+(s) - \dot{\boldsymbol{X}}_+(0)| = 0.$$

Furthermore, if $x_0 \in \Sigma$ and

$$0 \notin \nabla_p \mathcal{H}\left(x_0, u(x_0), \partial^+ u(x_0)\right),$$

then there exists $T_0 \in (0,T]$ such that $\boldsymbol{X}(t) \in \Sigma$ and $\dot{\boldsymbol{X}}(t) \neq 0$ when $t \in (0,T_0]$.

Remark 3. Suppose that $\Omega = \mathbb{R}^N$ where $N > 1$. It remains an open problem whether $\boldsymbol{X}(t) \in \Sigma$ for all $t \in [0,\infty)$.

4. The concept of strong characteristics

We define strong characteristics as forward regular broken characteristics in the following precise meaning.

Definition 6. A Lipschitz continuous arc $\boldsymbol{x}\colon [0,T) \to \Omega$ is called a *strong characteristic* for $(u,\mathcal{H})$ if

- the right derivative $\dot{\boldsymbol{x}}_+(t)$ exists at every $t \in [0,T)$;
- $\dot{\boldsymbol{x}}_+(\cdot)$ is right-continuous in $[0,T)$; and
- the equation

$$\dot{\boldsymbol{x}}_+(t) = \nabla_p \mathcal{H}\left(\boldsymbol{x}(t), u(\boldsymbol{x}(t)), \partial^+_{\mathcal{H}} u(\boldsymbol{x}(t))\right) \tag{15}$$

  holds true for every $t \in [0,T)$.

Theorem 4 (Cannarsa–Yu [29])**.** *Let (A), (B0) and (B1) be fulfilled. Suppose that a* unique *generalized characteristic for $(u,\mathcal{H})$ issues from every point in Ω. Then, each generalized characteristic $\boldsymbol{x}\colon [0,T) \to \Omega$ is, in fact, a strong characteristic.*

5. Time-dependent HJ

The classical Hamilton–Jacobi equation

(16) $$S_t + H(t, x, \nabla S) = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n,$$

(17) $$\lim_{t \downarrow 0} S(t, x) = S_0(x) \quad \text{in } \mathbb{R}^n.$$

Definition 7. Let S be a viscosity solution of (16). A *generalized characteristic* is a locally Lipschitz continuous curve $\boldsymbol{X} : I \to \mathbb{R}^n$ such that the differential inclusion

(18) $$\dot{\boldsymbol{X}}(t) \in \operatorname{co} \nabla_p H(t, \boldsymbol{X}(t), \nabla^+ S(t, \boldsymbol{X}(t)))$$

holds for almost all $t \in I$.

The *singular set* of S is defined as the set Σ of points where S fails to be differentiable. A basic result on the propagation of singularities reads:

Theorem 5 (Shock curves)**.** *Suppose that $H \in C^1$ and $p \mapsto H(t, x, p)$ is strictly convex for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$. Let S be a locally semiconcave viscosity solution of (16). Assume that $(t_0, x_0) \in \Sigma$. Then there exist $t_1 > t_0$ and a generalized characteristic $\boldsymbol{X}$ in the time interval $[t_0, t_1)$ such that $\boldsymbol{X}(t_0) = x_0$ and $(t, \boldsymbol{X}(t)) \in \Sigma$ for all $t \in [t_0, t_1)$.*

6. Prerequisites

The Hamiltonian H appearing in (16) is the Legendre–Fenchel transform of a Lagrangian L.

6.1. Lagrangian action and the value function. We define the Lagrangian *action* functional by

$$J(\boldsymbol{x};0,t) = S_0(\boldsymbol{x}(0)) + \int_0^t L(s,\boldsymbol{x}(s),\dot{\boldsymbol{x}}(s))\,ds, \tag{19}$$

for absolutely continuous arcs $\boldsymbol{x} \in \mathrm{AC}([0,t];\mathbb{R}^n)$. We consider the set of admissible arcs ending at (t,x), i.e.,

$$\mathcal{A}(t,x) = \{\boldsymbol{x} \in \mathrm{AC}([0,t];\mathbb{R}^n) \colon \boldsymbol{x}(t) = x\}.$$

Then, the *value function* is defined by

$$S(t,x) = \inf_{\boldsymbol{x}\in\mathcal{A}(t,x)} J(\boldsymbol{x};0,t), \qquad (t,x) \in Q. \tag{20}$$

We shall say that a curve $\boldsymbol{x}$ is an *action minimizer* in the interval $[t_0,t_1] \subset [0,\infty)$ if

$$\begin{aligned} S(t,\boldsymbol{x}(t)) &= J(\boldsymbol{x};t_0,t) \\ &:= S(t_0,\boldsymbol{x}(t_0)) + \int_{t_0}^t L(s,\boldsymbol{x}(s),\dot{\boldsymbol{x}}(s))\,ds \end{aligned}$$

for all times $t \in [t_0,t_1]$.

6.2. **Viscosity solution.**

Definition 8 (Viscosity solution)**.** A *viscosity subsolution* of (16) is an upper semicontinuous function $S\colon Q \to \mathbb{R}$ such that

$$\omega + H(t, x, p) \leq 0 \qquad \text{whenever } (\omega, p) \in \partial^+ S(t, x),$$

while a *viscosity supersolution* of (16) is a lower semicontinuous function S on Q such that

$$\omega + H(t, x, p) \geq 0 \qquad \text{whenever } (\omega, p) \in \partial^- S(t, x).$$

A *viscosity solution* is simultaneously a viscosity subsolution and a viscosity supersolution.

6.3. **Singular and conjugate points.** We repeat the definition of the singular set.

Definition 9. The *singular set* of S consists of the points in Q where S is not differentiable. It is denoted by Σ.

Assuming (A), (B) and (C) with $S_0 \in C^2$, let us denote by $(\boldsymbol{x}(t,y), \boldsymbol{p}(t,y))$ the solution of Hamilton's system

$$\dot{\boldsymbol{x}} = \nabla_p H(t, \boldsymbol{x}, \boldsymbol{p}),$$
$$\dot{\boldsymbol{p}} = -\nabla_x H(t, \boldsymbol{x}, \boldsymbol{p}),$$

subject to the initial condition $(\boldsymbol{x}(0), \boldsymbol{p}(0)) = (y, \nabla S_0(y))$. For any terminal point (t,x), each minimizer for the variational problem (20) has the form $\boldsymbol{x}(\cdot, y)$ for some $y \in \mathbb{R}^n$ (and $\boldsymbol{x}(t,y) = x$).

Definition 10. A point $(t,x) \in Q$ is called *conjugate* for the variational problem (20) if there exists a point $y \in \mathbb{R}^n$ such that $\boldsymbol{x}(t,y) = x$, $\boldsymbol{x}(\cdot, y)$ is a minimizer of (20), and $\det \nabla_y \boldsymbol{x}(t,y) = 0$. The set of conjugate points is denoted by Γ.

6.4. **An extract from the foundation of the calculus of variations.** In the following lemmas it is assumed that (A), (B) and (C) are fulfilled.

Lemma 1. *The value function S is locally semiconcave in Q and a viscosity solution of (16). The initial condition (17) is satisfied pointwise. If S is extended to $\overline{Q}$ by $S(0,\cdot) = S_0$, then S becomes lower semicontinuous on $\overline{Q}$.*

Lemma 2. *Let $\boldsymbol{x}$ be a minimizing arc of (20) for a given point $(t,x) \in Q$. Then $\boldsymbol{x} \in C^2([0,t])$ and $(s,\boldsymbol{x}(s)) \notin \Sigma$ for every $s \in (0,t)$. Moreover, $(t,x) \notin \Sigma$ if and only if the minimizer of (20) for (t,x) is unique.*

Lemma 3. *Two distinct minimizers may intersect only at their endpoints: if $\boldsymbol{x}_j \in \mathcal{A}(t_j,x_j)$ minimizes (20) for the terminal point $(t_j,x_j) \in Q$, $j = 1,2$, if $\boldsymbol{x}_1 \not\equiv \boldsymbol{x}_2$ on $[0,\min(t_1,t_2)]$, and if $\boldsymbol{x}_1(\tau) = \boldsymbol{x}_2(\tau)$, then either $\tau = 0$ or $\tau = t_1 = t_2$.*

Lemma 4. *Suppose in addition that $S_0 \in C^2$. Then $\overline{\Sigma} = \Sigma \cup \Gamma$ and $S \in C^2(Q \setminus \overline{\Sigma})$. Moreover, if $\boldsymbol{x}$ is a minimizing arc for (20) for a given point $(t,x) \in Q$, then $(s,\boldsymbol{x}(s)) \notin \overline{\Sigma}$ for every $s \in (0,t)$.*

Lemma 5. *Uniform limits of minimizers of (20) are again minimizers: if $\boldsymbol{x}_j \in \mathcal{A}(t_j,x_j)$ minimizes (20) for the terminal point $(t_j,x_j) \in Q$, if $(t_j,x_j) \to (\bar{t},\bar{x}) \in Q$, and if $\boldsymbol{x}_j \to \boldsymbol{x}$ uniformly, then $\boldsymbol{x}$ minimizes (20) for the terminal point $(\bar{t},\bar{x}) \in Q$.*

We next present a compactness property of minimizers.

Lemma 6. *If $\boldsymbol{x}_j \in \mathcal{A}(t_j,x_j)$ minimizes (20) for the terminal point $(t_j,x_j) \in Q$ and $(t_j,x_j) \to (\bar{t},\bar{x}) \in Q$, then a subsequence of $\boldsymbol{x}_j$ converges uniformly to $\boldsymbol{x}$, a minimizer of (20) for the terminal point $(\bar{t},\bar{x})$. If $(\bar{t},\bar{x}) \notin \Sigma$, then the full sequence $\boldsymbol{x}_j$ converges to $\boldsymbol{x}$, the unique minimizer of (20) for the terminal point $(\bar{t},\bar{x})$.*

7. A variational approach to singular dynamics

Our approach is to study the difference E between the Lagrangian action J and the value function S along a given arc $\boldsymbol{X}(t)$. The derivative dE/dt is represented in terms of a nonnegative function Y. The admissible velocity arises from minimizing $v \mapsto Y(t, x, v)$.

7.1. **Excess action.**

Definition 11. The *excess action* for a locally Lipschitz continuous curve $\boldsymbol{X} \colon [t_0, \infty) \to \mathbb{R}^n$, $t_0 \geq 0$, is defined by

$$E(\boldsymbol{X}; t_0, t) = S(t_0, \boldsymbol{X}(t_0)) + \int_{t_0}^{t} L(s, \boldsymbol{X}(s), \dot{\boldsymbol{X}}(s))\, ds - S(t, \boldsymbol{X}(t)) \quad \text{for all } t \in [t_0, \infty). \tag{21}$$

The functional E vanishes along a minimizing trajectory $\boldsymbol{x}$ for (20). Indeed, if $\boldsymbol{x}$ is a minimizer in the time interval $[0, T]$, then

$$S(t, \boldsymbol{x}(t)) = S(t_0, \boldsymbol{x}(t_0)) + \int_{t_0}^{t} L(s, \boldsymbol{x}(s), \dot{\boldsymbol{x}}(s))\, ds$$

and hence $E(\boldsymbol{x}; t_0, t) = 0$ whenever $0 \leq t_0 \leq t \leq T$. In a sense, the excess action measures how far a curve $\boldsymbol{X}$ is from being action minimizing.

The following proposition contains the dynamic programming principle.

Proposition 1. *Assume that (A), (B) and (C) are satisfied. Let* $t_0 \geq 0$ *and let* $\boldsymbol{X}\colon [t_0, \infty) \to \mathbb{R}^n$ *be locally Lipschitz continuous. If* $t_0 = 0$ *it is also assumed that* $S(t_0, \boldsymbol{X}(t_0)) = S_0(\boldsymbol{X}(0)) < \infty$*. Then the following assertions hold true.*

(i) $E(\boldsymbol{X}; t_0, t) \geq 0$ *for all* $t \geq t_0$.

(ii) $E(\boldsymbol{X}; t_0, t_2) = E(\boldsymbol{X}; t_0, t_1) + E(\boldsymbol{X}; t_1, t_2)$ *whenever* $t_0 \leq t_1 \leq t_2$.

(iii) *The function* $t \mapsto E(\boldsymbol{X}; t_0, t)$ *is nondecreasing and locally Lipschitz continuous (when* $t > 0$ *if* $t_0 = 0$*).*

(iv) *For any* $t_0 < t_1$, $E(\boldsymbol{X}; t_0, t_1) = 0$ *if and only if* $\boldsymbol{X}$ *is an action minimizer in* $[t_0, t_1]$*. In this case,* $\boldsymbol{X}$ *extends uniquely to an action minimizing arc* $\boldsymbol{x}$ *on* $[0, t_1]$.

We shall proceed by a heuristic reasoning. Let $(t_0, x_0) \in \Sigma$; this occurs precisely when there are several minimizing curves for (t_0, x_0), issuing from different locations yet bringing the same value of action to (t_0, x_0). In other words, in this case the Bolza problem (20) admits two or more minimizing arcs, which are necessarily extremals or classical characteristics. Once these characteristics cross, at (t_0, x_0), they cease to be minimizing arcs. Instead, there emanates from (t_0, x_0) a singular trajectory, i.e., a trajectory $\boldsymbol{X}$ with $\boldsymbol{X}(t_0) = x_0$ such that $(t, \boldsymbol{X}(t)) \in \Sigma$ for times $t \in [t_0, t_1)$.

It seems plausible that a singular trajectory can be singled out as the locally Lipschitz continuous curve $\boldsymbol{X}$ with $\boldsymbol{X}(t_0) = x_0$ that minimizes the rate of change of

$$E(t) := E(\boldsymbol{X}; t_0, t), \qquad t \in [t_0, \infty).$$

Indeed, although it cannot be an action minimizer, a singular trajectory should keep its excess action, measured by $E(t)$, growing as slowly as possible.

Proposition 2. *Under conditions (A), (B) and (C), let* $\boldsymbol{X}\colon [t_0, t_1) \to \mathbb{R}^n$, $t_0 > 0$, *be locally Lipschitz continuous. Then, at every* $t \in [t_0, t_1)$ *where the right derivative* $\dot{\boldsymbol{X}}_+(t)$ *exists, we have*

$$\frac{d^+}{dt} S(t, \boldsymbol{X}(t)) = \min_{(\omega,p)\in\partial^+ S(t,\boldsymbol{X}(t))} (\omega + p \cdot \dot{\boldsymbol{X}}_+(t)). \tag{22}$$

Moreover,

$$E(t) = \int_{t_0}^{t} Y(s, \boldsymbol{X}(s), \dot{\boldsymbol{X}}_+(s))\, ds \tag{23}$$

for all $t \in [t_0, t_1)$ *where* $E(t) = E(\boldsymbol{X}; t_0, t)$. *Hence*

$$\dot{E}_+(t) = Y(t, \boldsymbol{X}(t), \dot{\boldsymbol{X}}_+(t)) \quad \textit{for a.e. } t \in [t_0, t_1). \tag{24}$$

Here

$$Y(t, x, v) := L(t, x, v) - \min_{(\omega,p)\in\partial^+ S(t,x)} (\omega + p \cdot v)$$

for all $(t, x, v) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$; or,

$$Y(t, x, v) = L(t, x, v) + \max_{p\in\nabla^+ S(t,x)} (H(t, x, p) - p \cdot v),$$

$$\mathcal{P}(t, x, v) := \operatorname*{arg\,max}_{p\in\nabla^+ S(t,x)} (H(t, x, p) - p \cdot v).$$

Lemma 7. *Assume (A), (B) and (C). For every* $(t, x) \in Q$, $v \mapsto Y(t, x, v)$ *is a locally uniformly convex, superlinear, nonnegative function.*

7.2. **Admissible velocity and momentum.** In view of (24), to find arcs $\boldsymbol{X}$ along which the excess action grows as slowly as possible one should choose the forward velocity, if possible, by minimizing $v \mapsto Y(t, \boldsymbol{X}(t), v)$. This observation motivates the following definition.

Definition 12. Suppose that (A), (B) and (C) are met. For each $(t, x) \in Q$ the *admissible velocity* $v^*(t, x)$ is defined as the unique minimum point of $v \mapsto Y(t, x, v)$. The *admissible momentum* $p^*(t, x)$ is then $p^*(t, x) = \nabla_v L(t, x, v^*(t, x))$.

The admissible momentum p^* is a specific selection of $\nabla^+ S$ and $v^*(t, x) = \nabla_p H(t, x, p^*(t, x))$.

The importance of Y for detecting singular points stems from the observation that $v \mapsto Y(t, x, v)$ assumes the value 0 exactly when $(t, x) \notin \Sigma$:

$$Y(t, x, v^*(t, x)) \begin{cases} > 0 & \text{if } (t, x) \in \Sigma, \\ = 0 & \text{if } (t, x) \notin \Sigma. \end{cases}$$

7.3. **A minimization problem for the admissible momentum.** In the previous subsection, the admissible velocity $v^*(t,x)$ was defined by the optimality condition

$$Y(t,x,v) = \max_{p\in\nabla^+ S(t,x)} (L(t,x,v) + H(t,x,p) - p\cdot v) \to \min$$

with respect to $v \in \mathbb{R}^n$. However, $p^*(t,x)$ is more directly constructible through the minimization of $\omega + H(t,x,p)$ over $(\omega,p) \in \partial^+ S(t,x)$. To this end, fix (t,x) and consider the problem:

$$\text{Minimize} \quad \omega + H(t,x,p) \quad \text{over } (\omega,p) \in \partial^+ S(t,x). \tag{25}$$

Proposition 3. *Assume conditions (A), (B) and (C). The admissible momentum and velocity can then for any fixed* $(t,x) \in Q$ *be obtained as follows. The constrained convex minimization problem (25) possesses a unique solution* $(\omega^*(t,x), p^*(t,x)) \in \partial^+ S(t,x)$ *where* $p^*(t,x) \in \nabla^+ S(t,x)$ *is the admissible momentum at* (t,x)*. The admissible velocity is given by* $v^*(t,x) = \nabla_p H(t,x,p^*(t,x))$*; moreover,*

$$\begin{aligned}\min_{v\in\mathbb{R}^n} Y(t,x,v) &= Y(t,x,v^*(t,x)) \\ &= -(\omega^*(t,x) + H(t,x,p^*(t,x))) \\ &= -\min_{(\omega,p)\in\partial^+ S(t,x)} (\omega + H(t,x,p)).\end{aligned}$$

8. Strong characteristics

Definition 13. A locally Lipschitz arc $\boldsymbol{x}\colon [t_0, T_0) \to \mathbb{R}^n$, $t_0 > 0$, is called a *strong characteristic* if it is right differentiable and such that

$$\dot{\boldsymbol{x}}_+(t) = v^*(t, \boldsymbol{x}(t)) \quad \text{for each } t \in [t_0, T_0).$$

In view of Proposition 2, the derivative of $E(t) = E(\boldsymbol{x}; t_0, t)$, which is given by $Y(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}_+(t))$ almost everywhere, is minimal along a strong characteristic in the sense that

$$Y(t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}_+(t)) = \min_{v \in \mathbb{R}^n} Y(t, \boldsymbol{x}(t), v).$$

In this precise meaning, strong characteristics are trajectories along which the excess action grows at the slowest pace possible.

Remark 4. By Corollary 3.4 in [29], if generalized characteristics are uniquely determined by their initial data, then they are, in fact, strong characteristics. However, this uniqueness property fails in general [29, 45]; yet, it is fulfilled in the one-dimensional case ($n = 1$) as well as for the specific Hamiltonian $H(t, x, p) = W(t, x) + \langle p, A(t, x)p \rangle$ regardless of the dimension.

Proposition 4. *Assuming (A), (B) and (C), let $\boldsymbol{x}\colon [t_0, T_0) \to \mathbb{R}^n$ be a strong characteristic. Then*

$$\begin{aligned}\frac{d^+}{dt} S(t, \boldsymbol{x}(t)) &= \omega^*(t, \boldsymbol{x}(t)) + p^*(t, \boldsymbol{x}(t)) \cdot v^*(t, \boldsymbol{x}(t)) \\ &= \min_{(\omega, p) \in \partial^+ S(t, \boldsymbol{x}(t))} (\omega + p \cdot v^*(t, \boldsymbol{x}(t))) \quad \textit{for all } t \in [t_0, T_0),\end{aligned}$$

while

$$\begin{aligned}\dot{E}_+(t) = \frac{d^+}{dt} E(\boldsymbol{x}; t_0, t) &= \min_{v \in \mathbb{R}^n} Y(t, \boldsymbol{x}(t), v) \\ &= Y(t, \boldsymbol{x}(t), v^*(t, \boldsymbol{x}(t))).\end{aligned}$$

8.1. **Existence of strong characteristics.** To establish existence, the viscous parabolic regularization of (16)–(17) may be utilized:

$$S_t + H(t, x, \nabla S) - \mu \Delta S = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n,$$
$$S(0, x) = S_0(x) \quad \text{in } \mathbb{R}^n,$$

(a) $\{S^\mu\}$ is locally equi-bounded, equi-Lipschitz continuous as well as equi-semiconcave;
(b) As $\mu \downarrow 0$, a subsequence of $\{S^\mu\}$ converges locally uniformly to the value function S.

The regularization S^μ may be replaced by any other smooth approximating family enjoying properties (a) and (b). One may employ a specific mollification [29, Lemma 2.1] satisfying

$$\lim_{\varepsilon \downarrow 0} DS_\varepsilon(t_0, x_0) = (\omega^*(t_0, x_0), p^*(t_0, x_0)).$$

Theorem 6. *Assume condition (A). Let S be a viscosity solution of (16) which is semiconcave in $\overline{\Omega}$ where Ω is an open, convex, bounded, nonempty subset of $(0, \infty) \times \mathbb{R}^n$. For any point $(t_0, x_0) \in \Omega$ there exist a $T_0 > t_0$ and a locally Lipschitz continuous and right differentiable arc $\boldsymbol{x} \colon [t_0, T_0) \to \mathbb{R}^n$ whose graph is contained in Ω such that*

(i) $\boldsymbol{x}(t_0) = x_0$ *and* $\dot{\boldsymbol{x}}_+(t) = v^*(t, \boldsymbol{x}(t))$ *for each* $t \in [t_0, T_0)$;
(ii) *the right derivative* $\dot{\boldsymbol{x}}_+(t)$ *is right-continuous at* $t = t_0$; *and*
(iii) *if* $(t_0, x_0) \in \Sigma$ *then, for some* $t_1 \in (t_0, T_0]$, $(t, \boldsymbol{x}(t))$ *remains in* Σ *when* $t \in [t_0, t_1)$.

Assertion (i) says that $\boldsymbol{x}$ is a strong characteristic starting from (t_0, x_0). When $(t, \boldsymbol{x}(t))$ hits $\overline{\Sigma}$, then $\boldsymbol{x}(t)$ ceases to minimize the Lagrangian action (19), yet keeps its excess action growing as slowly as possible moving inside $\overline{\Sigma}$. Part (iii) confirms that singularities propagate along $\boldsymbol{x}$ forward in time t.

8.2. **Propagation of singularities.** The following assertions have been proved quite generally:

(a) If $(t_0, \boldsymbol{x}(t_0)) \in \overline{\Sigma}$, then $(t, \boldsymbol{x}(t)) \in \overline{\Sigma}$ for all later times $t \in [t_0, \infty)$;

(b) If $(t_0, \boldsymbol{x}(t_0)) \in \Sigma$, then $(t, \boldsymbol{x}(t)) \in \Sigma$ at least in some forward time interval $[t_0, t_1) \subseteq [t_0, \infty)$.

However, for a general Hamilton–Jacobi equation, $S_t + H(t, x, \nabla S) = 0$ in Q_T, $n \geq 2$, $H \in C^2$, satisfying conditions ensuring a well-behaved associated variational problem, it remains an open problem whether singularities propagate indefinitely, i.e., whether the following condition is fulfilled for every strong characteristic $\boldsymbol{x} \colon [t_0, \infty) \to \mathbb{R}^n$:

(c) If $(t_0, \boldsymbol{x}(t_0)) \in \Sigma$, then $(t, \boldsymbol{x}(t)) \in \Sigma$ for all $t \in [t_0, \infty)$.

Remark 5. Assertion (a) fails when H is nonsmooth as attested by the viscosity solution

$$S(t, x) = -a(t)|x|, \qquad C^1(\mathbb{R}) \ni a(t) = \begin{cases} (t-1)^2 & \text{if } t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

of

$$S_t + |\nabla S|^2 - 2\sqrt{a(t)}|x| - (a(t))^2 = 0.$$

Indeed,

$$\Sigma = \{(t, x) \colon 0 < t < 1,\ x = 0\}.$$

We recall Albano's theorem on propagation in $\overline{\Sigma}$.

Lemma 8. *Given* $(\bar{t}, \bar{x}) \in Q \setminus \overline{\Sigma}$, *let* $\boldsymbol{x}$ *be the unique minimizer of (20) for the terminal point* $(\bar{t}, \bar{x})$. *Then* $(\tau, \boldsymbol{x}(\tau)) \notin \overline{\Sigma}$ *for all* $\tau \in (0, \bar{t}]$.

Proof. Choose $s \in (0, \bar{t})$ arbitrarily. Let $B = B_\varepsilon(\bar{t}, \bar{x}) \subset Q$ be an open ball around $(\bar{t}, \bar{x})$ whose radius $\varepsilon > 0$ is so small that B does not intersect $\overline{\Sigma}$ and $\varepsilon < \bar{t} - s$. For each $(t, x) \in B$, (20) has a unique minimizer $\boldsymbol{\xi}(\cdot; t, x)$. Consider the map

$$G_s \colon B \to Q, \qquad B \ni (t, x) \mapsto (t - s, \boldsymbol{\xi}(t - s; t, x)) \in Q,$$

amounting to time-reversal, by time s, along the field of minimizers. Then G_s is injective and continuous owing to Lemmas 3 and 6. By the Domain Invariance Theorem of Brouwer, the range of G_s is open. Since $(\tau, \boldsymbol{\xi}(\tau; t, x))$ is non-singular for each $\tau \in (0, t)$, the range of G_s is a subset of $Q \setminus \Sigma$. In particular, the point $(\bar{t} - s, \boldsymbol{\xi}(\bar{t} - s; \bar{t}, \bar{x}))$ has an open neighborhood in which S is differentiable. We may conclude since s was arbitrarily chosen in $(0, \bar{t})$. □

Lemma 9. *Let conditions (A), (B), and (C) and be fulfilled. Assume that* $\boldsymbol{X} \colon [t_0, t_1] \to \mathbb{R}^n$, $0 < t_0 < t_1$, *is a generalized characteristic such that* $(t_1, \boldsymbol{X}(t_1)) \notin \overline{\Sigma}$. *Let* $\boldsymbol{x} \colon [0, t_1] \to \mathbb{R}^n$ *be the unique minimizer of (20) for the point* $(t_1, \boldsymbol{X}(t_1))$, *i.e.,*

$$S(t_1, \boldsymbol{X}(t_1)) = J(\boldsymbol{x}; 0, t_1), \qquad \boldsymbol{x}(t_1) = \boldsymbol{X}(t_1),$$

where J *is the action functional (19). Then* $(t, \boldsymbol{x}(t)) \notin \overline{\Sigma}$ *when* $t \in (0, t_1]$ *and* $\boldsymbol{X}(t) = \boldsymbol{x}(t)$ *for all* $t \in [t_0, t_1]$. *In particular,* $(t, \boldsymbol{X}(t)) = (t, \boldsymbol{x}(t)) \notin \overline{\Sigma}$ *whenever* $t \in [t_0, t_1]$.

Theorem 7. *Let conditions (A), (B), and (C) be fulfilled. Suppose that* $\boldsymbol{X} \colon [t_0, t_1) \to \mathbb{R}^n$ *is a generalized characteristic such that* $(t_0, \boldsymbol{X}(t_0)) \in \overline{\Sigma}$. *Then* $(t, \boldsymbol{X}(t)) \in \overline{\Sigma}$ *when* $t \in [t_0, t_1)$.

Proof. Suppose that $(\tau, \boldsymbol{X}(\tau))) \notin \overline{\Sigma}$ for some $\tau \in [t_0, t_1)$. By Lemma 9, $(t, \boldsymbol{X}(t)) \notin \overline{\Sigma}$ for all $t \in [t_0, \tau]$ which violates the hypothesis that $(t_0, \boldsymbol{X}(t_0)) \in \overline{\Sigma}$. □

Theorem 8. *Suppose that conditions (A), (B), and (C) are fulfilled. Suppose that S has no singularities in $\{t_1\} \times \overline{B}(x_1, R)$. Then S has in fact no singularities in the cone*

$$K = \{(t, x) \in (0, \infty) \times \mathbb{R}^n \colon t \le t_1,\ C(t_1 - t) + |x - x_1| \le R\}.$$

Proof. Since S is locally Lipschitz continuous there exists a constant $C \ge 0$ such that $|\dot{\boldsymbol{x}}(s)| \le C$ for all $s \in [t_0, t]$ and each minimizing arc $\boldsymbol{x}$ for $S(t, x)$ where the terminal point (t, x) lies in $[t_0, t_1] \times \overline{B}(x_1, R)$.

Fix an arbitrary point $(t_0, x_0) \in K$ with $t_0 < t_1$ and consider the mapping $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x) = x_0 + x - \boldsymbol{\xi}(t_0; t_1, x), \qquad x \in \mathbb{R}^n.$$

We notice that Φ is continuous, that

$$\Phi(x) - x_0 = \boldsymbol{\xi}(t_1; t_1, x) - \boldsymbol{\xi}(t_0; t_1, x),$$

and that, thus,

$$\begin{aligned} |\Phi(x) - x_1| &\le |\Phi(x) - x_0| + |x_0 - x_1| \\ &\le C(t_1 - t_0) + |x_0 - x_1| \le R \quad \text{for all} \quad x \in \overline{B}(x_1, R), \end{aligned}$$

since, first, the minimizing arc $t \mapsto \boldsymbol{\xi}(t; t_1, x)$ is Lipschitz continuous with rate C when $t \in [t_0, t_1]$ and, secondly, $(t_0, x_0) \in K$. It ensues that Φ maps the closed ball $\overline{B}(x_1, R)$ into itself. By Brouwer's fixed point theorem, there exists $x_* \in \overline{B}(x_1, R)$ such that $\Phi(x_*) = x_*$ and thus

$$x_0 = \boldsymbol{\xi}(t_0; t_1, x_*). \tag{26}$$

Equation (26) tells us that the minimizing curve $(t, \boldsymbol{\xi}(t; t_1, x_*))$, $t \in [0, t_1]$, passes through the point (t_0, x_0). Hence, (t_0, x_0) cannot be a singular point because the only points along a minimizing arc that may be singular are the endpoints. □

The theorem states that if Σ does not meet the base of a cone, then Σ is actually disjoint from the entire cone.

In the one-dimensional case, forward generalized characteristics are uniquely determined by their initial data. Hence, generalized and strong characteristics coincide [29]. We next recall Dafermos' result that a point moving in Σ forward in time along a generalized (broken) characteristic cannot escape Σ when $n = 1$ [32, Thm. 4.2].

Lemma 10 (Dafermos [32])**.** *Let $n = 1$. Assume that (A), (B) and (C) are fulfilled. Suppose that $\boldsymbol{x}\colon [t_0, \infty) \to \mathbb{R}^n$ is a strong characteristic whose initial point $(t_0, \boldsymbol{x}(t_0))$ is singular. Then $(t, \boldsymbol{x}(t)) \in \Sigma$ for all $t \in [t_0, \infty)$.*

Proof. Let $\boldsymbol{\xi}_-(\cdot)$ and $\boldsymbol{\xi}_+(\cdot)$ denote the minimal and maximal backward generalized characteristics through $(t_0, \boldsymbol{x}(t_0))$, respectively. By Theorem 3.2 of [32], $(t, \boldsymbol{\xi}_\pm(t)) \notin \Sigma$ for almost all $t \in (0, t_0)$. Then $\boldsymbol{\xi}_-(\cdot)$ and $\boldsymbol{\xi}_+(\cdot)$ action minimizers for the terminal point $(t_0, \boldsymbol{x}(t_0))$ (actually, the minimal and the maximal minimizer, respectively). Since $(t_0, \boldsymbol{x}(t_0))$ is a singular point, $\boldsymbol{\xi}_-(\cdot)$ and $\boldsymbol{\xi}_+(\cdot)$ must be distinct and hence cannot intersect in the time interval $(0, t_0)$. Fix any $s \in (t_0, t_1)$ and consider the minimal, $\boldsymbol{x}_-(\cdot)$, and the maximal, $\boldsymbol{x}_+(\cdot)$, backward characteristic emanating from $(s, \boldsymbol{x}(s))$. Since the curve given by $\boldsymbol{\xi}_-(t)$ when $t \in [0, t_0]$ and by $\boldsymbol{x}(t)$ when $t \in (t_0, s]$ is a generalized characteristic, the minimality of the generalized characteristic $\boldsymbol{x}_-(\cdot)$ on $[0, s]$ implies that $\boldsymbol{x}_-(t) \leq \boldsymbol{\xi}_-(t)$ when $t \in [0, t_0]$. Similarly, $\boldsymbol{x}_+(t) \geq \boldsymbol{\xi}_+(t)$ when $t \in [0, t_0]$. It follows that

$$\boldsymbol{x}_-(t) \leq \boldsymbol{\xi}_-(t) < \boldsymbol{\xi}_+(t) \leq \boldsymbol{x}_+(t)$$

when $t \in (0, t_0)$ and hence that $(s, \boldsymbol{x}(s)) \in \Sigma$ since $\boldsymbol{x}_\pm(\cdot)$ are action minimizers for the point $(s, \boldsymbol{x}(s))$. $\square$

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