

- The general relation for the specific heat is $C_v = \tau \left(\frac{\partial \sigma}{\partial \tau} \right)_v$
 - in case of the conduction electrons we have $C_v = \gamma \tau$ these two relations combine to give $\gamma \tau = \tau \left(\frac{\partial \sigma}{\partial \tau} \right)_v$ leading to $\frac{\partial \sigma}{\partial \tau} = \gamma = \text{constant}$. and hence integrating to $\sigma \propto \tau + \text{'new constant'}$ where the 'new constant' is zero as the the entropy is zero at temperature absolute zero. If the temperature increases from $\tau = 200\text{K}$ to 800K the entropy σ will increase by a factor **4**.
 - In the case of the electro magnetic field we have that the energy density is $u \propto \tau^4$ (Stefan–Boltzmann T^4 law) and hence we have for the specific heat $C_v \propto \tau^3$ (Note the similarity to phonons at low temperature the Debye T^3 law). As in a) we arrive at $\frac{\partial \sigma}{\partial \tau} \propto \tau^2$ and hence $\sigma \propto \tau^3$. If the temperature is raised from 500K to 2000K the entropy σ will increase by a factor of $\left(\frac{2000}{500}\right)^3 = 64$ that is a factor of **64**.
- Use the relation $\frac{1}{\tau} = \frac{\partial \sigma}{\partial E}$. $\sigma = \frac{c^3 A}{4G\hbar} = \frac{c^3 4\pi R_s^2}{4G\hbar} = \frac{c^3 4\pi 4G^2 M^2}{4G\hbar c^4} = \frac{4\pi G E^2}{\hbar c^5}$. Now we can evaluate for the temperature.
 - $\frac{1}{\tau} = \frac{\partial \sigma}{\partial E} = \frac{8\pi G M}{\hbar c^3}$, an alternative route is to use $\tau = \frac{\partial E}{\partial \sigma}$ this will give the same result.
 - $U = \frac{3}{2}\tau = 1\text{ eV}$ will give ratio $\frac{18e^{13.6/3^2\tau}}{8e^{13.6/2^2\tau}} = \frac{18}{8}e^{20.4(1/9-1/4)} = \frac{18}{8}e^{-2.833} = 0.1323 \approx 0.13$
- For a molecule the heat capacity C_v is given by $C_v = \left(\frac{\partial U}{\partial T} \right)_V$. The partition function is $Z = 1 + 3e^{-\epsilon/\tau}$. The energy is $U = \frac{3\epsilon e^{-\epsilon/\tau}}{Z}$

$$C_v = \left(\frac{\partial U}{\partial \tau} \right)_V = \frac{3\left(\frac{\epsilon}{\tau}\right)^2 e^{\epsilon/\tau}}{(3 + e^{\epsilon/\tau})^2} \quad (1)$$

Now let $x = \epsilon/\tau$

$$C_v = \frac{3x^2 e^x}{(3 + e^x)^2} \quad (2)$$

Taking the derivative $\frac{\partial C_v}{\partial x}$ we arrive at an equation for the maximum of C_v as $3(2 + x) + (2 - x)e^x = 0$. The maximum is at about $x \approx 2.845$, which gives $C_v = 1.023$ and using the correct units $C_v = 1.023k_B$.

- The partition sum is given by: $Z = \sum_{n_1=0, n_2=0}^{\infty} e^{-(n_1+n_2+1.0)\hbar\omega/k_B T} = \sum_{n=0}^{\infty} g(n)e^{-(n+1.0)\hbar\omega/k_B T}$, where $g(n)$ is the degeneracy of the energy levels and $n = n_1 + n_2$. There are two ways to evaluate this sum. One simple and one more elaborate. Only the simple solution is presented here. The sum for Z can be done as a product of two separate independent geometric sums.

$Z = \sum_{n_1=0, n_2=0}^{\infty} e^{-(n_1+n_2+1.0)\hbar\omega/k_B T} = \sum_{n_1=0}^{\infty} e^{-(n_1+0.5)\hbar\omega/k_B T} \cdot \sum_{n_2=0}^{\infty} e^{-(n_2+0.5)\hbar\omega/k_B T} =$
 $\left(\sum_{n=0}^{\infty} e^{-(n+0.5)\hbar\omega/k_B T}\right)^2 = e^{-\hbar\omega/k_B T} \left(\sum_{n=0}^{\infty} e^{-n\hbar\omega/k_B T}\right)^2 = e^{-\hbar\omega/k_B T} \left(\frac{1}{1-e^{-\hbar\omega/k_B T}}\right)^2$ and we arrive at the following for the partition function Z :

$$Z = \left(\frac{1}{e^{+\hbar\omega/2k_B T} - e^{-\hbar\omega/2k_B T}}\right)^2 = \text{or} = e^{-\hbar\omega/k_B T} \left(\frac{1}{1 - e^{-\hbar\omega/k_B T}}\right)^2.$$

- b) There is one state of the lower energy and two states with the next higher energy. The probability to find the oscillator in a state of energy is proportional to the Boltzmann factor, we arrive at the following equation. $1e^{-1,0\hbar\omega/k_B T} = 2e^{-2,0\hbar\omega/k_B T}$ and $e^{1\hbar\omega/k_B T} = 2$ which evaluates to $T = \frac{1\hbar\omega}{k_B \ln 2}$ or if you prefer τ , $\tau = \frac{1\hbar\omega}{\ln 2}$, which is equally correct.
- c) The partition sum at this specific temperature is given by: ($k_B T = \tau = \frac{1\hbar\omega}{\ln 2}$) we arrive at the following

$$Z = \left(\frac{1}{e^{+\frac{\ln 2}{2}} - e^{-\frac{\ln 2}{2}}}\right)^2 = \left(\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}}\right)^2$$

We continue with the calculation of the probability: (we may choose any of the two energies as their probabilities will be equal at the temperature in question: ($1e^{-1,0\hbar\omega/k_B T} = 2e^{-2,0\hbar\omega/k_B T}$))

$$P = e^{-\hbar\omega/k_B T} / \left(\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}}\right)^2 = 2^{-1} \left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^2 =$$

$$\left(1 - \frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$$

The probability to be in a state of one of these energies is $P = \frac{1}{4} = 0.25$

5. See also problem 10.5 in Kittel and Kroemer **a:** $F = -N_f \epsilon_0 + N_g \tau \left(\ln \frac{N_g}{V n_Q} - 1\right)$ **b:** $N_g = V n_Q e^{-\epsilon_0/\tau}$ **c:** Make a figure of $\ln p$ as a function of $1/T$. The slope of the straight line is $-\frac{\epsilon_0}{k_B}$ which gives an energy $\epsilon_0 = 0.53\text{eV}$.